

RESEARCH ARTICLE

WILEY

Stabilization for degenerate equations with drift and small singular term

Genni Fragnelli¹  | Dimitri Mugnai²  | Amine Sbai^{3,4} 

¹Department of Information Engineering and Mathematics, Siena University, Siena, Italy

²Department of Ecology and Biology, Tuscia University, Largo dell'Università, Viterbo, Italy

³Faculty of Sciences and Technology, MISI Laboratory, Hassan First University of Settat, Settat, Morocco

⁴Department of Applied Mathematic, University of Granada, Granada, Spain

Correspondence

Dimitri Mugnai, Department of Ecology and Biology, Tuscia University, Largo dell'Università, 01100 Viterbo, Italy.
 Email: dimitri.mugnai@unitus.it

Communicated by: C. Ji

Funding information

This study was funded by Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni, Ministero dell'Università e della Ricerca, and Ministero degli Affari Esteri e della Cooperazione Internazionale.

We consider a degenerate/singular wave equation in lone dimension, with drift and in presence of a leading operator that is not in divergence form. We impose a homogeneous Dirichlet boundary condition where the degeneracy occurs and a boundary damping at the other endpoint. We provide some conditions for the uniform exponential decay of solutions for the associated Cauchy problem.

KEYWORDS

degenerate wave equation, drift, exponential decay, singular potentials, stabilization

MSC CLASSIFICATION

35L80, 35L81, 93D23, 93D15

1 | INTRODUCTION

This paper is devoted to study the stabilization of degenerate equations of waves type with drift and with a small singular perturbation through a linear boundary feedback. To be more precise, we consider the following problem:

$$\begin{cases} y_{tt} - a(x)y_{xx} - b(x)y_x - \frac{\lambda}{d(x)}y = 0, & (t, x) \in Q, \\ y_t(t, 1) + \eta y_x(t, 1) + \beta y(t, 1) = 0, & t \in (0, +\infty), \\ y(t, 0) = 0, & t > 0, \\ y(0, x) = y_0(x), \quad y_t(0, x) = y_1(x), & x \in (0, 1), \end{cases} \quad (1.1)$$

This is an open access article under the terms of the Creative Commons Attribution License, which permits use, distribution and reproduction in any medium, provided the original work is properly cited.

© 2024 The Author(s). Mathematical Methods in the Applied Sciences published by John Wiley & Sons Ltd.

where $Q = (0, +\infty) \times (0, 1)$, $a, b \in C^0[0, 1]$, with $a > 0$ on $(0, 1]$, $a(0) = 0$ and $\frac{b}{a} \in L^1(0, 1)$.¹ In the boundary term, we take $\beta \geq 0$ and η is the well-known absolutely continuous weight function defined by

$$\eta(x) := \exp \left\{ \int_{\frac{1}{2}}^x \frac{b(s)}{a(s)} ds \right\}, \quad x \in [0, 1], \quad (1.2)$$

introduced by Feller several years ago [1]. As usual (see, e.g. [2]), we find that $\eta \in C^0[0, 1] \cap C^1(0, 1]$ is a strictly positive function and it can be extended to a function of class $C^1[0, 1]$ when b degenerates at 0 not slower than a (for instance, if $a(x) = x^K$ and $b(x) = x^h$, $h \geq K$).

The fact that $a(0) = 0$ introduces a lack of ellipticity in the spatial operator and for this reason the equation is *degenerate*. Degenerate equations have attracted great attention in the last decades, since they appear in several applied contexts, such as physics [3], biology [4, 5], mathematical finance [6], climatology [7], and image processing ([8]). In particular, degenerate parabolic equations have been object of an intensive research in the last two decades [9–18], mainly from the point of view of null controllability, in general obtained by new Carleman estimates. Problems in presence of both a degeneracy and a singularity have been considered, as well [19–25].

A much less studied situation is the one of waves-type equations admitting a degeneracy term [26–32] or a singular term [33, 34], for which the authors prove controllability or (exponential) stability results.

As far as we know, for waves-type equations admitting simultaneously degeneracy and singularity, only controllability problems have been faced (see the recent papers [35–37]), while nothing has been done for stability. For this reason, in this paper, we focus on such a problem, proving that (1.1) permits boundary stabilization, provided that the singular term has a small coefficient (see Theorem 3.2). Hence, we may regard this result as a perturbation of the related one in Fragnelli et al. [36]. However, the presence of the singular term $\frac{1}{d}u$ introduces several difficulties, which let us treat only the case of a function d with *weak degeneracy*, according to the definition below.

Definition 1. A function g is weakly degenerate at 0, (WD) for short, if $g \in C^0[0, 1] \cap C^1(0, 1]$ is such that $g(0) = 0$, $g > 0$ on $(0, 1]$ and, if

$$\sup_{x \in (0, 1]} \frac{x |g'(x)|}{g(x)} = K_g, \quad (1.3)$$

then $K_g \in (0, 1)$.

Definition 2. A function g is strongly degenerate at 0, (SD) for short, if $g \in C^1[0, 1]$ is such that $g(0) = 0$, $g > 0$ on $(0, 1]$, and in (1.3), we have $K_g \in [1, 2)$.

Referring to Theorem 3.2 for the precise assumptions and statement, a flavor of our exponential stability result is the following:

If λ is small and a, b are not too degenerate, then
the energy of the solution to (1.1) converges exponentially to 0 as time diverges.

A case covered by our study is the one in which the equation in (1.1) is

$$y_{tt} - x^\alpha y_{xx} - \mu x^\beta y_x - \frac{\lambda}{x^\gamma} y = 0.$$

In this case, we have exponential stability provided that

$$\beta > 0, \quad \beta > \alpha - 1, \quad 2 - \alpha - 2\gamma > 2\mu \quad \text{and} \quad \lambda \text{ small enough.}$$

The paper is organized as follows. In Section 2, we give the functional setting and some technical tools that we will use in the rest of the paper, together with the existence of solutions. In Section 3, we introduce the energy associated to a solution of the problem and, by a multiplier method, we show that it decays exponentially as time diverges. In the Appendix A, we prove the existence theorem stated in Section 2.

¹If $a(x) = x^K$, $K > 0$, we can consider $b(x) = x^h$ for any $h > K - 1$.

2 | PRELIMINARY RESULTS AND WELL POSEDNESS

In this section, we introduce the functional setting needed to face our problem, and then we give some preliminary results about the function spaces we treat, and finally, we prove the well posedness result.

Following Fragnelli et al. [36], we start assuming a very modest requirement.

Hypothesis 1. *The functions $a, b, d \in C^0[0, 1]$ are such that*

1. $\frac{b}{a} \in L^1(0, 1)$,
2. $a(0) = d(0) = 0$, $a, d > 0$ on $(0, 1]$,
3. *there exist $K_1, K_2 \in (0, 2)$ such that the functions*

$$x \mapsto \frac{x^{K_1}}{a(x)} \quad (2.4)$$

and

$$x \mapsto \frac{x^{K_2}}{d(x)} \quad (2.5)$$

are nondecreasing in a right neighborhood of $x = 0$.

Remark 1. If a is (WD) or (SD), then (1.3) implies that the function

$$x \mapsto \frac{x^\gamma}{a(x)} \text{ is nondecreasing in } (0, 1] \text{ for all } \gamma \geq K_a. \quad (2.6)$$

In particular, (2.4) holds and

$$\left| \frac{x^\gamma b(x)}{a(x)} \right| \leq \tilde{M} \quad (2.7)$$

for all $\gamma \geq K_a$, where

$$\tilde{M} := \frac{\|b\|_{L^\infty(0,1)}}{a(1)}. \quad (2.8)$$

Moreover,

$$\lim_{x \rightarrow 0} \frac{x^\gamma}{a(x)} = 0 \quad (2.9)$$

for all $\gamma > K_a$.

Analogously, if d is (WD) or (SD), then

$$x \mapsto \frac{x^\gamma}{d(x)} \text{ is nondecreasing in } (0, 1] \quad (2.10)$$

for all $\gamma \geq K_d$ and

$$\lim_{x \rightarrow 0} \frac{x^\gamma}{d(x)} = 0 \quad (2.11)$$

for all $\gamma > K_d$.

Remark 2. Concerning the notation, we will use K_1 (K_2 , respectively) when Hypothesis 1 holds, and K_a (K_d , respectively) when the global degeneracy conditions (WD) or (SD) hold true.

Now set

$$\sigma(x) := \frac{a(x)}{\eta(x)}, \quad x \in [0, 1],$$

and consider, as in Fragnelli and Mugnai [38], the following Hilbert spaces with the related inner products:

$$L^2_{\frac{1}{\sigma}}(0, 1) := \left\{ u \in L^2(0, 1) \mid \int_0^1 \frac{u^2}{\sigma} dx < \infty \right\}, \quad \langle u, v \rangle_{\frac{1}{\sigma}} := \int_0^1 uv \frac{1}{\sigma} dx$$

for every $u, v \in L^2_{\frac{1}{\sigma}}(0, 1)$ and

$$H^1_{\frac{1}{\sigma}}(0, 1) := L^2_{\frac{1}{\sigma}}(0, 1) \cap H^1(0, 1), \quad \langle u, v \rangle_{1, \frac{1}{\sigma}} := \langle u, v \rangle_{\frac{1}{\sigma}} + \int_0^1 \eta u' v' dx$$

for every $u, v \in H^1_{\frac{1}{\sigma}}(0, 1)$. The previous inner products obviously induce the related norms

$$\|u\|_{\frac{1}{\sigma}}^2 = \int_0^1 \frac{u^2}{\sigma} dx \quad \text{and} \quad \|u\|_{1, \frac{1}{\sigma}}^2 = \|u\|_{\frac{1}{\sigma}}^2 + \int_0^1 \eta (u')^2 dx.$$

Moreover, consider the space

$$H^1_{\frac{1}{\sigma}, 0}(0, 1) := \left\{ u \in H^1_{\frac{1}{\sigma}}(0, 1) : u(0) = 0 \right\}$$

endowed with the same inner product and norm of $H^1_{\frac{1}{\sigma}}(0, 1)$. Clearly, thanks to (2.4), and (2.5), if $u \in H^1_{\frac{1}{\sigma}, 0}(0, 1)$, then one can estimate the two integrals $\int_0^1 \frac{u^2}{\sigma} dx$ and $\int_0^1 \frac{u^2}{\sigma d} dx$ with the norm of u' . Indeed, as in Boutaayamou et al. [30] and in Fragnelli et al. [36], the following Hardy–Poincaré-type inequalities hold.

Proposition 1. Assume Hypothesis 1. Then, there exists $C > 0$ such that for all $u \in H^1_{\frac{1}{\sigma}, 0}(0, 1)$,

$$\int_0^1 u^2 \frac{1}{\sigma} dx \leq C \int_0^1 (u')^2 dx. \quad (2.12)$$

Moreover, if $K_1 + K_2 \leq 2$ and $u \in H^1_{\frac{1}{\sigma}, 0}(0, 1)$ then $\frac{u}{\sqrt{\sigma d}} \in L^2(0, 1)$ and there exists a positive constant $C > 0$ such that

$$\int_0^1 \frac{u^2}{\sigma d} dx \leq C \int_0^1 \eta (u'(x))^2 dx. \quad (2.13)$$

Let

$$\tilde{C}_{HP} \text{ and } C_{HP} \text{ be the best constants in (2.12) and (2.13),} \quad (2.14)$$

respectively. Clearly, $\tilde{C}_{HP} \leq C_{HP} \max_{[0,1]} d \max_{[0,1]} \eta$. Moreover, under an additional assumption on λ , one can consider a different norm on $H^1_{\frac{1}{\sigma}, 0}(0, 1)$. Assume the following:

Hypothesis 2. The constant $\lambda \in \mathbb{R} \setminus \{0\}$ is such that

$$\lambda < \frac{1}{C_{HP}}. \quad (2.15)$$

Observe that it is not restrictive to assume $\lambda \neq 0$, since in Fragnelli and Mugnai [38], the case $\lambda = 0$ is considered.

If $K_1 + K_2 \leq 2$ and if Hypotheses 1 and 2 hold, then one can consider on $H^1_{\frac{1}{\sigma}, 0}(0, 1)$ also the inner product

$$\langle u, v \rangle_1 := \langle u, v \rangle_{\frac{1}{\sigma}} + \int_0^1 \eta u' v' dx - \lambda \int_0^1 \frac{uv}{\sigma d},$$

which induces the norm

$$\|u\|_1^2 := \|u\|_{\frac{1}{\sigma}}^2 + \int_0^1 \eta (u')^2 dx - \lambda \int_0^1 \frac{u^2}{\sigma d}$$

for all $u, v \in H_{\sigma}^1(0, 1)$. In particular, setting

$$\|u\|_{1,\circ}^2 := \int_0^1 (u')^2 dx$$

and

$$\|u\|_{1,\bullet}^2 := \int_0^1 \eta(u')^2 dx$$

for all $u \in H_{\sigma}^1(0, 1)$, one has the following equivalence thanks to Proposition 1:

Corollary 1. Assume Hypotheses 1 and 2. If $K_1 + K_2 \leq 2$, then the norms $\|\cdot\|_{1,\frac{1}{\sigma}}$, $\|\cdot\|_1$, $\|\cdot\|_{1,\circ}$ and $\|\cdot\|_{1,\bullet}$ are equivalent in $H_{\sigma}^1(0, 1)$.

Now define the operator

$$Ay := ay'' + by' = \sigma(\eta y')' \quad (2.16)$$

for all $y \in D(A)$, where $D(A)$ is the Hilbert space

$$D(A) = H_{\sigma}^2(0, 1) := \left\{ u \in H_{\sigma}^1(0, 1) \mid Au \in L_{\sigma}^2(0, 1) \right\},$$

with inner product

$$\langle u, v \rangle_{2,\frac{1}{\sigma}} := \langle u, v \rangle_{1,\frac{1}{\sigma}} + \langle Au, Av \rangle_{\frac{1}{\sigma}}$$

or

$$\langle u, v \rangle_2 := \langle u, v \rangle_1 + \langle Au, Av \rangle_{\frac{1}{\sigma}}.$$

In the rest of the paper, the next lemma is crucial; since its proof is similar to the one of Fragnelli et al. [36, lemma 3.2], we omit it.

Lemma 1. Assume Hypothesis 1.

1. If $y \in D(A)$ and $u \in H_{\sigma}^1(0, 1)$, then $\lim_{x \rightarrow 0} u(x)y'(x) = 0$.
2. If $u \in H_{\sigma}^1(0, 1)$, then $\lim_{x \rightarrow 0} \frac{x}{a} u^2(x) = 0$.
3. If $u \in H_{\sigma}^1(0, 1)$ and $K_1 + K_2 < 2$, then $\lim_{x \rightarrow 0} \frac{x}{ad} u^2(x) = 0$.
4. If $u \in H_{\sigma}^1(0, 1)$, then $\lim_{x \rightarrow 0} \frac{x^2}{ad} u^2(x) = 0$.
5. If $u \in D(A)$, then $\lim_{x \rightarrow 0} x^2(u'(x))^2 = 0$.
6. If $u \in D(A)$, $K_1 \leq 1$, then $\lim_{x \rightarrow 0} x(u'(x))^2 = 0$.
7. If $u \in D(A)$, $K_1 > 1$ and $\frac{xb}{a} \in L^{\infty}(0, 1)$, then $\lim_{x \rightarrow 0} x(u'(x))^2 = 0$.

Thanks to the first point of the previous lemma one has the following integration by parts

Lemma 2 (Fragnelli and Mugnai [38, lemma 2.1]). Assume Hypothesis 1. If $u \in H_{\sigma}^2(0, 1)$ and $v \in H_{\sigma}^1(0, 1)$, then

$$\langle Au, v \rangle_{\frac{1}{\sigma}} = - \int_0^1 \eta u' v' dx + [\eta u' v](1).$$

Now, define

$$A_{\lambda} u := Au + \lambda \frac{u}{d} \quad (2.17)$$

for all $u \in D(A_\lambda) := \left\{ u \in H_{\frac{1}{\sigma},0}^2(0,1) : A_\lambda u \in L_{\frac{1}{\sigma}}^2(0,1) \right\}$. Clearly, $D(A_\lambda) \subseteq D(A)$; anyway, if $K_2 \leq 1$, we have $\frac{u}{d} \in L^2(0,1)$, and so

$$D(A) = D(A_\lambda);$$

however, in general, such an inequality does not hold, unless $K_1 + 2K_2 \leq 2$ [36]. Moreover, by Lemma 2, one has that if Hypothesis 1 is satisfied, then

$$\langle A_\lambda u, v \rangle_{\frac{1}{\sigma}} = - \int_0^1 \eta u' v' dx + \lambda \int_0^1 \frac{uv}{\sigma d} dx + [\eta u' v](1) \quad (2.18)$$

for all $u \in H_{\frac{1}{\sigma},0}^2(0,1)$ and $v \in H_{\frac{1}{\sigma},0}^1(0,1)$. Thus, thanks to the new operator $(A_\lambda, D(A_\lambda))$, one can prove the next existence and regularity results, whose proof, being standard, is postponed to the Appendix A.

Theorem 2.1. Assume Hypotheses 1 and 2 with $K_1 + 2K_2 \leq 2$. If $(y_0, y^1) \in H_{\frac{1}{\sigma},0}^1(0,1) \times L_{\frac{1}{\sigma}}^2(0,1)$, then there exists a unique mild solution

$$y \in C^1\left([0, +\infty); L_{\frac{1}{\sigma}}^2(0,1)\right) \cap C\left([0, +\infty); H_{\frac{1}{\sigma},0}^1(0,1)\right)$$

of (1.1) that depends continuously on the initial data (y_0, y_1) . Moreover, if $(y_0, y_1) \in H_{\frac{1}{\sigma},0}^2(0,1) \times H_{\frac{1}{\sigma},0}^1(0,1)$, then the solution y is classical, in the sense that

$$y \in C^2\left([0, +\infty); L_{\frac{1}{\sigma}}^2(0,1)\right) \cap C^1\left([0, +\infty); H_{\frac{1}{\sigma},0}^1(0,1)\right) \cap C([0, +\infty); D(A_\lambda))$$

and the equation of (1.1) holds for all $t \geq 0$.

Observe that the assumption $K_1 + 2K_2 \leq 2$ implies $K_2 < 1$. This assumption is also crucial to prove the second part of the next variational result. The case that corresponds to a singular being strongly degenerate with $K_d \geq 1$ is an open problem.

The next preliminary result will be crucial for the following.

Proposition 2. Assume Hypotheses 1 and 2 and for $\beta \geq 0$ define

$$|||z|||_1^2 := \int_0^1 \eta(z')^2 dx - \lambda \int_0^1 \frac{z^2}{\sigma d} dx + \beta z^2(1)$$

for all $z \in H_{\frac{1}{\sigma},0}^1(0,1)$. Then the norms $||| \cdot |||_1$, $\| \cdot \|_{1,\frac{1}{\sigma}}$, $\| \cdot \|_1$, $\| \cdot \|_{1,\bullet}$ and $\| \cdot \|_{1,\circ}$ are equivalent in $H_{\frac{1}{\sigma},0}^1(0,1)$. Moreover, for every $\gamma \in \mathbb{R}$, the variational problem

$$\int_0^1 \eta z' \phi' dx - \lambda \int_0^1 \frac{z\phi}{\sigma d} + \beta z(1)\phi(1) = \gamma \phi(1), \quad \forall \phi \in H_{\frac{1}{\sigma},0}^1(0,1) \quad (2.19)$$

admits a unique solution $Z \in H_{\frac{1}{\sigma},0}^1(0,1)$, which satisfies the estimates

$$|||Z|||_1^2 \leq \gamma^2 C_\lambda^2 \text{ and } \|Z\|_{\frac{1}{\sigma}}^2 \leq \left(\tilde{C}_{HP} + \max_{[0,1]} \eta \right) \gamma^2 C_\lambda^4, \quad (2.20)$$

where C_{HP} , $\tilde{C}_{HP}$ are the best Hardy–Poincaré constants from Proposition 1 and

$$C_\lambda := \begin{cases} \frac{1}{\sqrt{\min_{[0,1]} \eta}}, & \lambda \leq 0, \\ \frac{1}{\sqrt{\varepsilon \min_{[0,1]} \eta}}, & \lambda \in \left(0, \frac{1}{C_{HP}}\right), \end{cases} \quad (2.21)$$

where $\varepsilon \in (0, 1)$ is such that

$$\lambda = \frac{1 - \varepsilon}{C_{HP}}. \quad (2.22)$$

In addition, if $K_1 + 2K_2 \leq 2$, then $Z \in D(A_\lambda)$ and solves

$$\begin{cases} -A_\lambda Z = 0, \\ \eta Z'(1) + \beta Z(1) = \gamma. \end{cases} \quad (2.23)$$

Proof. To prove the equivalence among the norms, it is sufficient to prove that $||| \cdot |||_1$ and $|| \cdot ||_{1,\circ}$ are equivalent, thanks to Corollary 1. Now, if $\lambda \leq 0$, it is clear that for all $z \in H_{\frac{1}{\sigma},0}^1(0, 1)$

$$\|z\|_{1,\circ}^2 \leq \frac{1}{\min_{[0,1]}\eta} |||z|||_1^2.$$

If $\lambda \in \left(0, \frac{1}{C_{HP}}\right)$, we start fixing ε as indicated in (2.22). Hence, for all $z \in H_{\frac{1}{\sigma},0}^1(0, 1)$, one has

$$\int_0^1 \eta(z')^2 dx - \lambda \int_0^1 \frac{z^2}{\sigma d} dx \geq \varepsilon \int_0^1 \eta(z')^2 dx \quad (2.24)$$

and so

$$\|z\|_{1,\circ}^2 \leq \frac{1}{\varepsilon \min_{[0,1]}\eta} |||z|||_1^2$$

for all $z \in H_{\frac{1}{\sigma},0}^1(0, 1)$. In conclusion, we have

$$\|z\|_{1,\circ} \leq C_\lambda |||z|||_1, \quad (2.25)$$

where C_λ is as in (2.21). Now, we prove that there exists $C > 0$ such that

$$|||z|||_1^2 \leq C \|z\|_{1,\circ}^2$$

for all $z \in H_{\frac{1}{\sigma},0}^1(0, 1)$. First of all, observe that

$$|z(1)| = \left| \int_0^1 z'(t) dt \right| \leq \|z\|_{1,\circ} \quad (2.26)$$

for all $z \in H_{\frac{1}{\sigma},0}^1(0, 1)$. Now let $\lambda > 0$. Clearly, if $z \in H_{\frac{1}{\sigma},0}^1(0, 1)$, from (2.26), we get

$$|||z|||_1^2 \leq \left(\max_{[0,1]} \eta + \beta \right) \|z\|_{1,\circ}^2.$$

On the other hand, if $\lambda < 0$ and $z \in H_{\frac{1}{\sigma},0}^1(0, 1)$, by (2.26), we get

$$|||z|||_1^2 \leq (1 - \lambda C_{HP}) \max_{[0,1]} \eta \int_0^1 (z')^2 dx + \beta z^2(1) \leq ((1 - \lambda C_{HP}) \max_{[0,1]} \eta + \beta) \|z\|_{1,\circ}^2.$$

Thus, the two norms are equivalent.

Now, consider the bilinear and symmetric form $\Lambda : H_{\sigma,0}^1(0,1) \times H_{\sigma,0}^1(0,1) \rightarrow \mathbb{R}$, given by

$$\Lambda(z, \phi) := \int_0^1 \eta z' \phi' dx - \lambda \int_0^1 \frac{z\phi}{\sigma d} dx + \beta z(1)\phi(1)$$

for all $z, \phi \in H_{\sigma,0}^1(0,1)$. Clearly, Λ is coercive and continuous. Indeed, by Corollary 1

$$\Lambda(z, z) = \int_0^1 \eta (z')^2 dx - \lambda \int_0^1 \frac{z^2}{\sigma d} dx + \beta z^2(1) \geq \int_0^1 \eta (z')^2 dx - \lambda \int_0^1 \frac{z^2}{\sigma d} dx \geq \|z\|_{1,\bullet}^2, \quad (2.27)$$

if $\lambda \leq 0$, and

$$\Lambda(z, z) \geq (1 - \lambda C_{HP}) \|z\|_{1,\bullet}^2, \quad (2.28)$$

if $\lambda \in \left(0, \frac{1}{C_{HP}}\right)$. Moreover, by the Hölder inequality and (2.26),

$$|\Lambda(z, \phi)| \leq \int_0^1 \eta |z' \phi'| dx - \lambda \int_0^1 \left| \frac{z\phi}{\sigma d} \right| dx + \beta |z(1)| |\phi(1)| \leq \left((1 - \lambda C_{HP}) \max_{x \in [0,1]} \eta + \beta \right) \|z\|_{1,\circ} \|\phi\|_{1,\circ},$$

if $\lambda \leq 0$, and

$$|\Lambda(z, \phi)| \leq (2 \max_{x \in [0,1]} \eta + \beta) \|z\|_{1,\circ} \|\phi\|_{1,\circ},$$

if $\lambda \in \left(0, \frac{1}{C_{HP}}\right)$.

Now, consider the linear functional $\mathcal{L} : H_{\sigma,0}^1(0,1) \rightarrow \mathbb{R}$ defined as

$$\mathcal{L}(\phi) := \gamma \phi(1),$$

for every $\phi \in H_{\sigma,0}^1(0,1)$. Clearly, $\mathcal{L}$ is continuous and linear. Thus, by the Lax–Milgram theorem, there exists a unique solution $Z \in H_{\sigma,0}^1(0,1)$ of

$$\Lambda(Z, \phi) = \mathcal{L}(\phi) \quad (2.29)$$

for all $\phi \in H_{\sigma,0}^1(0,1)$. In particular,

$$\|Z\|_1^2 = \Lambda(Z, Z) = \int_0^1 \eta (Z')^2 dx - \lambda \int_0^1 \frac{Z^2}{\sigma d} dx + \beta Z^2(1) = \mathcal{L}(Z) = \gamma Z(1). \quad (2.30)$$

By (2.25) and (2.26), we have that for all $z \in H_{\sigma,0}^1(0,1)$

$$|z(1)| \leq \|z\|_{1,\circ} \leq C_\lambda \|z\|_1 \quad (2.31)$$

and by (2.30),

$$\|Z\|_1^2 = \gamma Z(1) \leq |\gamma| C_\lambda \|Z\|_1;$$

thus,

$$\|Z\|_1 \leq |\gamma| C_\lambda,$$

which proves the first part of (2.20). Moreover, by Proposition 1, (2.25) and (2.31), we have

$$\begin{aligned} \|Z\|_{\frac{1}{\sigma}}^2 &\leq \|Z\|_{1, \frac{1}{\sigma}}^2 \leq \left(\tilde{C}_{HP} + \max_{[0,1]} \eta \right) \|Z\|_{1,0}^2 \leq \left(\tilde{C}_{HP} + \max_{[0,1]} \eta \right) C_{\lambda}^2 \|Z\|_1^2 \\ &\leq \left(\tilde{C}_{HP} + \max_{[0,1]} \eta \right) \gamma^2 C_{\lambda}^4, \end{aligned}$$

thus concluding the validity of (2.20).

Now, assume $K_1 + 2K_2 \leq 2$. We will prove that $Z \in H_{\frac{1}{\sigma},0}^2(0,1)$ and solves (2.23). To this aim, we consider again (2.29). Since it is true for every $\phi \in H_{\frac{1}{\sigma},0}^1(0,1)$, it holds in particular for every $\phi \in C_c^\infty(0,1)$, so that

$$\int_0^1 \eta Z' \phi' - \lambda \int_0^1 \frac{Z\phi}{\sigma d} = 0 \quad \forall \phi \in C_c^\infty(0,1) \iff \int_0^1 \eta Z' \phi' = \lambda \int_0^1 \frac{Z\phi}{\sigma d} \quad \forall \phi \in C_c^\infty(0,1).$$

Hence,

$$(\eta Z')' = -\lambda \frac{Z}{\sigma d} \quad \text{a.e. in } (0,1) \iff -\sigma(\eta Z')' - \lambda \frac{Z}{d} = 0 \quad \text{a.e. in } (0,1)$$

and so $A_\lambda Z \in L_{\frac{1}{\sigma}}^2(0,1)$. Thanks to the assumption $K_1 + 2K_2 \leq 2$, one has that $\frac{Z}{d} \in L_{\frac{1}{\sigma}}^2(0,1)$. Hence, $AZ \in L_{\frac{1}{\sigma}}^2(0,1)$ and so $Z \in D(A_\lambda)$.

Finally, coming back to (2.29), we have by (2.18)

$$\int_0^1 \eta Z' \phi' dx - \lambda \int_0^1 \frac{Z\phi}{\sigma d} dx + \beta Z(1)\phi(1) = \gamma \phi(1) \iff (\eta Z' \phi)(1) + \beta Z(1)\phi(1) = \gamma \phi(1)$$

for all $\phi \in H_{\frac{1}{\sigma},0}^1(0,1)$. Thus, we obtain

$$(\eta Z')(1) + \beta Z(1) = \gamma;$$

hence, Z solves (2.23). $\square$

Remark 3. The previous result is similar to the one given in Fragnelli and Mugnai [38, proposition 2.2], but here, the presence of the singular term makes the situation more involved, and requires the condition $K_1 + 2K_2 \leq 2$.

3 | STABILITY RESULT

In this section, we prove the main result of this paper when a and d are (WD) or (SD). To this aim, if y is a mild solution of (1.1), we consider its energy given by

$$E_y(t) = \frac{1}{2} \left[\int_0^1 \left(\frac{1}{\sigma} y_t^2(t,x) + \eta y_x^2(t,x) - \frac{\lambda}{\sigma d} y^2(t,x) \right) dx + \beta y^2(t,1) \right], \quad t \geq 0,$$

and we prove that it decreases exponentially under suitable assumptions. In particular, we collect all the previous assumptions in the following one:

Hypothesis 3. Assume that

1. the functions a and d are (WD) or (SD),
2. $b \in C^0[0,1]$ is such that $\frac{b}{a} \in L^1(0,1)$,
3. $K_a + 2K_d \leq 2$,
4. if $K_a > 1$, then $\frac{xb}{a} \in L^\infty(0,1)$
5. $\lambda < \frac{1}{C_{HP}}$.

We remark that the condition $\frac{xb}{a} \in L^\infty(0, 1)$ is automatically verified if $K_a \leq 1$ by (2.7). Moreover, we recall that the condition $K_a + 2K_d \leq 2$ (which specifies the previous $K_1 + 2K_2 \leq 2$) implies that d cannot be (SD). Under the previous hypotheses, one can prove the next result

Theorem 3.1. Assume Hypothesis 3 and let y be a classical solution of (1.1). Then the energy is nonincreasing and

$$\frac{dE_y(t)}{dt} = -y_t(t, 1)^2, \quad t \geq 0.$$

Proof. By multiplying the equation by $\frac{y_t}{\sigma}$, integrating over $(0, 1)$ and using the boundary conditions, one has

$$0 = \frac{1}{2} \int_0^1 \frac{d}{dt} \left(\frac{y_t^2}{\sigma} \right) dx - \eta(1) y_x(t, 1) y_t(t, 1) + \int_0^1 \eta y_x y_{tx} dx - \int_0^1 \frac{\lambda}{\sigma d} y y_t dx.$$

Thus,

$$0 = \frac{1}{2} \frac{d}{dt} \left[\int_0^1 \left(\frac{y_t^2}{\sigma} + \eta y_x^2 - \frac{\lambda}{\sigma d} y^2 \right) dx + \beta y^2(t, 1) \right] + y_t^2(t, 1) = \frac{d}{dt} E_y(t) + y_t^2(t, 1).$$

Hence, the conclusion follows. $\square$

Now, our aim is to estimate the energy at time $t > 0$, $E_y(t)$, in terms of the energy at time $t = 0$, $E_y(0)$. To do that, we start with the following:

Proposition 3. Assume Hypothesis 3 and let y be a classical solution of (1.1). Then, for all $T > s > 0$

$$\begin{aligned} 0 &= 2 \int_0^1 \left[\frac{x y_x y_t}{\sigma} \right]_{t=s}^{t=T} dx - \frac{1}{\sigma(1)} \int_s^T y_t^2(t, 1) dt - \eta(1) \int_s^T y_x^2(t, 1) dt - \frac{\lambda}{\sigma(1)d(1)} \int_0^T y^2(t, 1) dt \\ &\quad - \int_{Q_s} x \eta \frac{b}{a} y_x^2 dx dt + \int_{Q_s} \left(1 - \frac{x(a' - b)}{a} \right) \frac{1}{\sigma} y_t^2 dx dt + \int_{Q_s} \eta y_x^2 dx dt \\ &\quad + \lambda \int_{Q_s} \left(1 - \frac{x(a' - b)}{a} - \frac{x d'}{d} \right) \frac{y^2}{\sigma d} dx dt, \end{aligned} \quad (3.32)$$

where $Q_s := (s, T) \times (0, 1)$.

Proof. Take any $T > s > 0$; then, multiplying the equation of (1.1) by $\frac{x y_x}{\sigma}$, integrating over Q_s and recalling (2.16), we have, after some integration by parts,

$$\begin{aligned} 0 &= \int_{Q_s} \frac{y_{tt} x y_x}{\sigma} dx dt - \int_{Q_s} x (\eta y_x)_x y_x dx dt - \lambda \int_{Q_s} \frac{x y y_x}{\sigma d} dx dt \\ &= \int_0^1 \left[\frac{x y_x y_t}{\sigma} \right]_{t=s}^{t=T} dx - \frac{1}{2} \int_s^T \left[\frac{x}{\sigma} y_t^2 \right]_{x=0}^{x=1} dt + \frac{1}{2} \int_{Q_s} \left(\frac{x}{\sigma} \right)' y_t^2 dx dt - \int_{Q_s} x \eta \frac{b}{a} y_x^2 dx dt \\ &\quad - \frac{1}{2} \int_s^T [x \eta y_x^2]_{x=0}^{x=1} dt + \frac{1}{2} \int_{Q_s} (x \eta)' y_x^2 dx dt - \frac{\lambda}{2} \int_s^T \left[\frac{x y^2}{\sigma d} \right]_{x=0}^{x=1} dt \\ &\quad + \frac{\lambda}{2} \int_{Q_s} \left(\frac{\sigma d - x(\sigma' d + \sigma d')}{(\sigma d)^2} \right) y^2 dx dt. \end{aligned}$$

Recalling the definitions of η and σ , we finally find

$$\begin{aligned}
 0 &= \int_0^1 \left[\frac{xy_x y_t}{\sigma} \right]_{t=s}^{t=T} dx - \frac{1}{2} \int_s^T \left[\frac{x}{\sigma} y_t^2 \right]_{x=0}^{x=1} dt - \frac{1}{2} \int_s^T [x\eta y_x^2]_{x=0}^{x=1} dt - \frac{\lambda}{2} \int_s^T \left[\frac{xy^2}{\sigma d} \right]_{x=0}^{x=1} dt \\
 &\quad - \frac{1}{2} \int_{Q_s} x\eta \frac{b}{a} y_x^2 dx dt + \frac{1}{2} \int_{Q_s} \left(1 - \frac{x(a'-b)}{a} \right) \frac{1}{\sigma} y_t^2 dx dt + \frac{1}{2} \int_{Q_s} \eta y_x^2 dx dt \\
 &\quad + \frac{\lambda}{2} \int_{Q_s} \left(1 - \frac{x(a'-b)}{a} - \frac{xd'}{d} \right) \frac{y^2}{\sigma d} dx dt.
 \end{aligned} \tag{3.33}$$

Moreover, using the boundary conditions and Lemma 1, we have that

$$\lim_{x \rightarrow 0} \frac{x}{\sigma} y_t^2(t, x) = \lim_{x \rightarrow 0} \frac{x}{a} \eta y_t^2(t, x) = 0,$$

$$\lim_{x \rightarrow 0} x\eta y_x^2(t, x) = 0$$

and

$$\lambda \int_0^T \left[y^2 \frac{x}{\sigma d} \right]_{x=0}^{x=1} = \frac{\lambda}{\sigma(1)d(1)} \int_0^T y^2(t, 1) dt.$$

Hence, (3.33) multiplied by 2 gives (3.32). $\square$

Proposition 4. Assume Hypothesis 3 and let y be a classical solution of (1.1). Then, for all $T > s > 0$, we have

$$\begin{aligned}
 (B.T.) &= \int_{Q_s} \left(1 - \frac{x(a'-b)}{a} + \frac{K_a}{2} \right) \frac{1}{\sigma} y_t^2 dx dt + \int_{Q_s} \left(1 - x\frac{b}{a} - \frac{K_a}{2} \right) \eta y_x^2 dx dt \\
 &\quad + \lambda \int_{Q_s} \left(1 - \frac{x(a'-b)}{a} - \frac{xd'}{d} + \frac{K_a}{2} \right) \frac{y^2}{\sigma d} dx dt,
 \end{aligned} \tag{3.34}$$

where

$$\begin{aligned}
 (B.T.) &= \int_0^1 \left[-2 \frac{xy_x y_t}{\sigma} + \frac{K_a}{2} \frac{yy_t}{\sigma} \right]_{t=s}^{t=T} dx \\
 &\quad + \int_s^T \left[\frac{1}{\sigma} y_t^2 + \eta y_x^2 - \frac{K_a}{2} \eta y_x y + \frac{\lambda}{\sigma d} y^2 \right] (t, 1) dt.
 \end{aligned} \tag{3.35}$$

Proof. Multiplying the equation in (1.1) by $\frac{y}{\sigma}$ and integrating over Q_s , we have

$$0 = \int_{Q_s} \left(-\frac{y_t^2}{\sigma} + \eta y_x^2 - \lambda \frac{y^2}{\sigma d} \right) dx dt + \int_0^1 \left[\frac{yy_t}{\sigma} \right]_{t=s}^{t=T} dx - \int_s^T [\eta y_x y]_{x=0}^{x=1} dt. \tag{3.36}$$

Using the boundary conditions and the fact that y is a classical solution of (1.1), one has

$$\int_s^T [\eta y_x y]_{x=0}^{x=1} dt = \int_s^T [\eta y_x y](t, 1) dt;$$

thus, multiplying (3.36) by $-\frac{K_a}{2}$, we obtain

$$0 = -\frac{K_a}{2} \int_{Q_s} \left(-\frac{y_t^2}{\sigma} + \eta y_x^2 - \lambda \frac{y^2}{\sigma d} \right) dx dt - \frac{K_a}{2} \int_0^1 \left[\frac{yy_t}{\sigma} \right]_{t=s}^{t=T} dx + \frac{K_a}{2} \int_s^T [\eta y_x y](t, 1) dt. \quad (3.37)$$

Summing (3.37) and (3.32), we get the claim. $\square$

Now, define

$$\mathbf{1}_\epsilon := \begin{cases} 1, & \lambda \leq 0, \\ \epsilon, & \lambda = \frac{1-\epsilon}{C_{HP}}. \end{cases} \quad (3.38)$$

The following result holds:

Proposition 5. Assume Hypothesis 3, $\beta \geq 0$ and let y be a classical solution of (1.1). Then, there exists $\mathbf{1}_\epsilon \in (0, 1]$, such that for any $T > s > 0$ and $\delta > 0$, we have

$$\begin{aligned} \int_s^T y^2(t, 1) dt &\leq \left(\frac{2}{\mathbf{1}_\epsilon} \left(1 + \frac{\tilde{C}_{HP} C_\lambda^2}{\mathbf{1}_\epsilon \min_{[0,1]}^2 \eta} \right) + \frac{1}{2\delta} + \frac{1}{2\delta} (\tilde{C}_{HP} + \max_{[0,1]} \eta) C_\lambda^4 \right) E_y(s) \\ &\quad + \frac{\delta}{\mathbf{1}_\epsilon} \left(1 + \frac{1}{\mathbf{1}_\epsilon \min_{[0,1]}^2 \eta} C_\lambda^2 \right) \int_s^T E_y(t) dt, \end{aligned} \quad (3.39)$$

where C_λ is as in (2.21).

Proof. The proof is similar to the one of proposition 3.3 in Fragnelli and Mugnai [38], so we will be sketchy. Fix $t \in [s, T]$, set $\gamma = y(t, 1)$ and let $z = z(t, \cdot)$ be the unique solution of

$$\int_0^1 \eta z' \phi' dx - \lambda \int_0^1 \frac{z\phi}{\sigma d} dx + \beta z(1)\phi(1) = \gamma \phi(1), \quad \forall \phi \in H_{\frac{1}{\sigma}, 0}^1(0, 1).$$

By Proposition 2, $z(t, \cdot) \in D(A_\lambda)$ for all t and solves

$$\begin{cases} A_\lambda z = 0, \\ \eta z_x(t, 1) + \beta z(t, 1) = \gamma. \end{cases} \quad (3.40)$$

As in Fragnelli and Mugnai [38], we multiply the equation in (1.1) by $\frac{z}{\sigma}$ and integrate over Q_s . Then, applying Lemma 1, we have (notice that $z_t \in L_{\frac{1}{\sigma}}^2(0, 1)$ by (2.20))

$$\int_0^1 \left[y_t \frac{z}{\sigma} \right]_{t=s}^{t=T} dx - \int_{Q_s} y_t \frac{z_t}{\sigma} dx dt - \lambda \int_{Q_s} \frac{y}{\sigma d} z dx dt = \int_s^T \eta y_x(t, 1) z(1) dt - \int_{Q_s} \eta y_x z_x dx dt. \quad (3.41)$$

Multiplying the equation in (3.40) by $\frac{y}{\sigma}$, integrating on Q_s and using the boundary conditions, one has

$$\int_s^T (\gamma - \beta z(t, 1)) y(t, 1) dt = \int_{Q_s} \eta z_x y_x dx dt - \lambda \int_{Q_s} \frac{y}{\sigma d} z dx dt.$$

Substituting in (3.41) and recalling that y solves (1.1), so that, in particular, $\eta y_x(t, 1) + \beta y(t, 1) = -y_t(t, 1)$, and $\gamma = y(t, 1)$, we have

$$\int_s^T y^2(t, 1) dt = \int_{Q_s}^T \frac{y_t z_t}{\sigma} dx dt - \int_s^T (y_t z)(t, 1) dt - \int_0^1 \left[\frac{y_t z}{\sigma} \right]_{t=s}^{t=T} dx. \quad (3.42)$$

Hence, to estimate $\int_s^T y^2(t, 1) dt$, we have to consider the right-hand-side (3.42). As a first step, observe that, if $\lambda \leq 0$, then

$$\frac{1}{2} \int_0^1 \left(\frac{1}{\sigma} y_t^2(t, x) + \eta y_x^2(t, x) \right) dx \leq E_y(t)$$

for all $t \in [s, T]$. If $\lambda \in \left(0, \frac{1}{C_{HP}}\right)$ and $\varepsilon \in (0, 1)$ is as in (2.22), we obtain (2.24), which implies that

$$\begin{aligned} 2E_y(t) &\geq \varepsilon \int_0^1 \eta y_x^2(t, x) dx + \int_0^1 \frac{y_t^2(t, x)}{\sigma(x)} dx + \beta y^2(t, 1) \\ &\geq \varepsilon \left(\int_0^1 \eta y_x^2(t, x) dx + \int_0^1 \frac{y_t^2(t, x)}{\sigma(x)} dx \right) + \beta y^2(t, 1); \end{aligned}$$

in particular,

$$\frac{1}{2} \int_0^1 \left(\frac{1}{\sigma} y_t^2(t, x) + \eta y_x^2(t, x) \right) dx \leq \frac{1}{\varepsilon} E_y(t)$$

for all $t \geq 0$.

Hence, for all considered λ and all $t > 0$, we have

$$\frac{1}{2} \int_0^1 \left(\frac{1}{\sigma} y_t^2(t, x) + \eta y_x^2(t, x) \right) dx \leq \frac{1}{\mathbf{I}_\varepsilon} E_y(t) \quad (3.43)$$

and

$$\frac{1}{2} y^2(t, 1) = \frac{1}{2} \left(\int_0^1 y_x(t, x) dx \right)^2 \leq \frac{1}{2} \int_0^1 y_x^2(t, x) dx \leq \frac{1}{2 \min_{[0,1]} \eta} \int_0^1 (\eta y_x^2)(t, x) dx \leq \frac{1}{\mathbf{I}_\varepsilon \min_{[0,1]} \eta} E_y(t). \quad (3.44)$$

Let us start estimating the third term in (3.42). To this aim, observe that

$$\int_0^1 (\eta z_x^2) \leq \frac{1}{\mathbf{I}_\varepsilon} |||z|||_1^2 \quad (3.45)$$

by (2.27) and (2.28). Hence, applying Proposition 1, (3.45), (2.20), (3.43), and (3.44), Theorem 3.1, and recalling that $\gamma = y(t, 1)$, we have

$$\begin{aligned} \int_0^1 \left| \frac{y_t z}{\sigma}(\tau, x) \right| dx &\leq \frac{1}{2} \int_0^1 \frac{y_t^2(\tau, x)}{\sigma} dx + \frac{1}{2} \int_0^1 \frac{z^2(\tau, x)}{\sigma} dx \\ &\leq \frac{1}{2} \int_0^1 \frac{y_t^2(\tau, x)}{\sigma} dx + \frac{1}{2 \min_{[0,1]} \eta} \tilde{C}_{HP} \int_0^1 (\eta z_x^2)(\tau, x) dx \\ &\leq \frac{1}{2} \int_0^1 \frac{y_t^2(\tau, x)}{\sigma} dx + \frac{\tilde{C}_{HP} C_\lambda^2}{\mathbf{I}_\epsilon^2 \min_{[0,1]} \eta} y^2(t, 1) \\ &\leq \frac{1}{\mathbf{I}_\epsilon} E_y(\tau) + \frac{\tilde{C}_{HP} C_\lambda^2}{\mathbf{I}_\epsilon^2 (\min_{[0,1]} \eta)^2} E_y(t) \\ &\leq \frac{1}{\mathbf{I}_\epsilon} \left(1 + \frac{\tilde{C}_{HP} C_\lambda^2}{\mathbf{I}_\epsilon (\min_{[0,1]} \eta)^2} \right) E_y(s) \end{aligned}$$

for all $\tau \in [s, T]$. Here C_λ is the constant defined in (2.21). Thus, again by Theorem 3.1, we have

$$\left| \int_0^1 \left[\frac{y_t z}{\sigma} \right]_{t=s}^{t=T} dx \right| \leq \frac{2}{\mathbf{I}_\epsilon} \left(1 + \frac{\tilde{C}_{HP} C_\lambda^2}{\mathbf{I}_\epsilon (\min_{[0,1]} \eta)^2} \right) E_y(s). \quad (3.46)$$

Now, consider the second term in (3.42). Then, for any $\delta > 0$, we have

$$\int_s^T |(y_t z)(t, 1)| dt \leq \frac{1}{2\delta} \int_s^T y_t^2(t, 1) dt + \frac{\delta}{2} \int_s^T z^2(t, 1) dt. \quad (3.47)$$

Starting as in (3.44), by (2.20), (3.45), and (3.44), we have

$$\begin{aligned} z^2(t, 1) &\leq \frac{1}{\min_{[0,1]} \eta} \int_0^1 (\eta z_x^2)(t, x) dx \leq \frac{1}{\mathbf{I}_\epsilon \min_{[0,1]} \eta} |||z|||_1^2 \\ &\leq \frac{1}{\mathbf{I}_\epsilon \min_{[0,1]} \eta} C_\lambda^2 y^2(t, 1) \leq \frac{2}{\mathbf{I}_\epsilon^2 \min_{[0,1]}^2 \eta} C_\lambda^2 E_y(t). \end{aligned}$$

Thus, by Theorem 3.1, we have

$$\begin{aligned} \int_s^T |(y_t z)(t, 1)| dt &\leq \frac{1}{2\delta} \int_s^T y_t^2(t, 1) dt + \frac{\delta}{2} \int_s^T z^2(t, 1) dt \\ &\leq -\frac{1}{2\delta} \int_s^T \frac{dE_y(t)}{dt} dt + \delta \frac{1}{\mathbf{I}_\epsilon^2 \min_{[0,1]}^2 \eta} C_\lambda^2 \int_s^T E_y(t) dt \\ &\leq \frac{E_y(s)}{2\delta} + \delta \frac{1}{\mathbf{I}_\epsilon^2 \min_{[0,1]}^2 \eta} C_\lambda^2 \int_s^T E_y(t) dt. \end{aligned} \quad (3.48)$$

Finally, we estimate $\int_{Q_s} \frac{y_t z_t}{\sigma} dx dt$. To this aim, consider again (3.40) and differentiate with respect to t . Then

$$\begin{cases} -\sigma(\eta z_{tx})_x - \lambda \frac{z_t}{d} = 0, \\ \eta z_{tx}(t, 1) + \beta z_t(t, 1) = \gamma_t = y_t(t, 1). \end{cases}$$

By (2.20), z_t satisfies the estimates

$$\|z_t\|_1^2 \leq C_\lambda^2 y_t^2(t, 1) \text{ and } \|z_t\|_{\frac{1}{\sigma}}^2 \leq \left(\tilde{C}_{HP} + \max_{[0,1]} \eta \right) C_\lambda^4 y_t^2(t, 1).$$

Thus, by (3.43) and Theorem 3.1, we find

$$\begin{aligned} \int_{Q_s} \left| \frac{y_t z_t}{\sigma} \right| dx dt &\leq \frac{\delta}{2} \int_{Q_s} \frac{y_t^2}{\sigma} dx dt + \frac{1}{2\delta} \int_{Q_s} \frac{z_t^2}{\sigma} dx dt \\ &\leq \frac{\delta}{\mathbf{1}_\epsilon} \int_s^T E_y(t) dt + \frac{1}{2\delta} \left(\tilde{C}_{HP} + \max_{[0,1]} \eta \right) C_\lambda^4 \int_s^T y_t^2(t, 1) dt \\ &= \frac{\delta}{\mathbf{1}_\epsilon} \int_s^T E_y(t) dt - \frac{1}{2\delta} \left(\tilde{C}_{HP} + \max_{[0,1]} \eta \right) C_\lambda^4 \int_s^T \frac{dE_y(t)}{dt} dt \\ &\leq \frac{\delta}{\mathbf{1}_\epsilon} \int_s^T E_y(t) dt + \frac{1}{2\delta} \left(\tilde{C}_{HP} + \max_{[0,1]} \eta \right) C_\lambda^4 E_y(s). \end{aligned} \quad (3.49)$$

Hence, going back to (3.42), by (3.49), (3.48), and (3.46), we get

$$\begin{aligned} \int_s^T y^2(t, 1) dt &\leq \left(\frac{2}{\mathbf{1}_\epsilon} + \frac{2}{\mathbf{1}_\epsilon^2} \frac{\tilde{C}_{HP} C_\lambda^2}{\min_{[0,1]}^2 \eta} + \frac{1}{2\delta} + \frac{1}{2\delta} (\tilde{C}_{HP} + \max_{[0,1]} \eta) C_\lambda^4 \right) E_y(s) \\ &\quad + \frac{\delta}{\mathbf{1}_\epsilon} \left(1 + \frac{1}{\mathbf{1}_\epsilon \min_{[0,1]}^2 \eta} C_\lambda^2 \right) \int_s^T E_y(t) dt, \end{aligned}$$

as claimed. $\square$

As in Fragnelli and Mugnai [38], we assume an additional hypothesis on functions a and b . Clearly, this assumption is stronger with respect to the one in Fragnelli and Mugnai [38] since here, we have also a singular term to manage.

Hypothesis 4. Hypothesis 3 holds and there exists $\varepsilon_0 > 0$ such that $(2 - K_a - 2K_d)a - 2x|b| \geq \varepsilon_0 a$ for every $x \in [0, 1]$.

Proposition 6. Assume Hypothesis 4 and let y be a classical solution of (1.1). Then, for any $T > s > 0$:

$$\begin{aligned} \frac{\varepsilon_0}{2} \int_{Q_s} \left(\frac{y_t^2}{\sigma} + \eta y_x^2 - \lambda \frac{y^2}{\sigma d} \right) dx dt &\leq \left(2\Theta + \frac{1}{\sigma(1)} + \frac{1}{\eta(1)} + \frac{\beta}{\eta(1)} + \frac{K_a}{4} \right) E_y(s) \\ &\quad + \left(\frac{\beta^2}{\eta(1)} + \frac{K_a \beta}{2} + \frac{\beta}{\eta(1)} + \frac{K_a}{4} + \frac{\lambda}{\sigma(1)d(1)} \right) \int_s^T y^2(t, 1) dt, \end{aligned} \quad (3.50)$$

if $\lambda \geq 0$, and

$$\begin{aligned} \frac{\varepsilon_0}{2} \int_{Q_s} \left(\frac{y_t^2}{\sigma} + \eta y_x^2 - \lambda \frac{y^2}{\sigma d} \right) dx dt \leq & \left(2\Theta + \frac{1}{\sigma(1)} + \frac{1}{\eta(1)} + \frac{\beta}{\eta(1)} + \frac{K_a}{4} \right) E_y(s) \\ & + \left(\frac{\beta^2}{\eta(1)} + \frac{K_a \beta}{2} + \frac{\beta}{\eta(1)} + \frac{K_a}{4} \right) \int_s^T y^2(t, 1) dt \\ & - 2\lambda C_{HP} \left(1 + \frac{3}{2} K_a + K_d + M \right) \int_s^T E_y(t) dt, \end{aligned} \quad (3.51)$$

if $\lambda < 0$. Here,

$$\Theta := \frac{2}{1_\varepsilon} \max \left\{ \frac{1}{a(1)} + \frac{K_a \tilde{C}_{HP}}{4 \min_{[0,1]} \eta}, 1 + \frac{K_a}{4} \right\}, \quad (3.52)$$

1_ε is given in (3.38) and

$$M := \begin{cases} \tilde{M}, & K_a \leq 1, \\ \left\| \frac{xb}{a} \right\|_{L^\infty(0,1)}, & K_a > 1, \end{cases}$$

where $\tilde{M}$ is defined in (2.8).

Proof. Thanks to Hypothesis 4, we have

$$\begin{aligned} 1 - \frac{x(a' - b)}{a} + \frac{K_a}{2} & \geq \frac{(2 - K_a - 2K_d)a + 2K_d a - 2x|b|}{2a} \geq \frac{\varepsilon_0}{2} + K_d > \frac{\varepsilon_0}{2}, \\ 1 - x \frac{b}{a} - \frac{K_a}{2} & \geq \frac{(2 - K_a - 2K_d)a + 2K_d a - 2x|b|}{2a} \geq \frac{\varepsilon_0}{2} + K_d > \frac{\varepsilon_0}{2}, \\ 1 - \frac{x(a' - b)}{a} - \frac{xd'}{d} + \frac{K_a}{2} & \geq \frac{(2 - 2K_a)a - 2x|b| + K_a a - 2K_d a}{2a} \geq \frac{\varepsilon_0}{2}. \end{aligned} \quad (3.53)$$

Now, we distinguish between the case $\lambda \geq 0$ and $\lambda < 0$.

Case $\lambda \geq 0$. In this case, the distributed terms in (3.34) can be estimated from below in the following way:

$$\begin{aligned} & \int_{Q_s} \left(1 - \frac{x(a' - b)}{a} + \frac{K_a}{2} \right) \frac{1}{\sigma} y_t^2 dx dt + \int_{Q_s} \left(1 - x \frac{b}{a} - \frac{K_a}{2} \right) \eta y_x^2 dx dt \\ & + \lambda \int_{Q_s} \left(1 - \frac{x(a' - b)}{a} - \frac{xd'}{d} + \frac{K_a}{2} \right) \frac{y^2}{\sigma d} dx dt \\ & \geq \frac{\varepsilon_0}{2} \int_{Q_s} \left(\frac{y_t^2}{\sigma} + \eta y_x^2 - \lambda \frac{y^2}{\sigma d} \right) dx dt. \end{aligned} \quad (3.54)$$

Now, we estimate the boundary terms in (3.35) from above. First of all, consider the term

$$\int_0^1 \left(-2x \frac{y_x y_t}{\sigma} + \frac{K_a}{2} \frac{y y_t}{\sigma} \right) (\tau, x) dx$$

for all $\tau \in [s, T]$. By (2.6), one has that $\frac{x^2}{a(x)} \leq \frac{1}{a(1)}$; using this fact together with Proposition 1 and (3.43), one has

$$\begin{aligned} & \int_0^1 \left(-2x \frac{y_x y_t}{\sigma} + \frac{K_a}{2} \frac{y y_t}{\sigma} \right) (\tau, x) dx \\ & \leq \int_0^1 \frac{x^2 y_x^2 \eta}{a} (\tau, x) dx + \int_0^1 \frac{y_t^2}{\sigma} (\tau, x) dx + \frac{K_a}{4} \int_0^1 \frac{y_t^2}{\sigma} (\tau, x) dx + \frac{K_a}{4} \int_0^1 \frac{y^2}{\sigma} (\tau, x) dx \\ & \leq \frac{1}{a(1)} \int_0^1 \eta y_x^2 (\tau, x) dx + \left(1 + \frac{K_a}{4} \right) \int_0^1 \frac{y_t^2}{\sigma} (\tau, x) dx + \frac{K_a \tilde{C}_{HP}}{4 \min_{[0,1]} \eta} \int_0^1 \eta y_x^2 (\tau, x) dx \\ & \leq \frac{2}{1_\epsilon} \max \left\{ \frac{1}{a(1)} + \frac{K_a \tilde{C}_{HP}}{4 \min_{[0,1]} \eta}, 1 + \frac{K_a}{4} \right\} E_y(\tau) \end{aligned}$$

for all $\tau \in [s, T]$. In particular,

$$\int_0^1 \left[\left(-2x \frac{y_x y_t}{\sigma} + \frac{K_a}{2} \frac{y y_t}{\sigma} \right) (\tau, x) dx \right]_{\tau=s}^{\tau=T} \leq 2\Theta E_y(s), \quad (3.55)$$

where Θ is defined in (3.52). By (3.34), (3.54) and the previous inequality, we get

$$\frac{\epsilon_0}{2} \int_{Q_s} \left(\frac{y_t^2}{\sigma} + \eta y_x^2 - \lambda \frac{y^2}{\sigma d} \right) dx dt \leq 2\Theta E_y(s) + \int_s^T \left[\frac{1}{\sigma} y_t^2 + \eta y_x^2 - \frac{K_a}{2} \eta y_x y + \frac{\lambda}{\sigma d} y^2 \right] (t, 1) dt. \quad (3.56)$$

As Fragnelli and Mugnai [38, equation (3.22)], one can prove that

$$\begin{aligned} & \int_s^T \left[\frac{1}{\sigma} y_t^2 + \eta y_x^2 - \frac{K_a}{2} \eta y_x y + \frac{\lambda}{\sigma d} y^2 \right] (t, 1) dt \leq \left(\frac{1}{\sigma(1)} + \frac{1}{\eta(1)} + \frac{\beta}{\eta(1)} + \frac{K_a}{4} \right) (E_y(s) - E_y(T)) \\ & \quad + \left(\frac{\beta^2}{\eta(1)} + \frac{K_a \beta}{2} + \frac{\beta}{\eta(1)} + \frac{K_a}{4} + \frac{\lambda}{\sigma(1)d(1)} \right) \int_s^T y^2(t, 1) dt. \end{aligned} \quad (3.57)$$

Hence, by (3.55), (3.56), and (3.57), we have

$$\begin{aligned} & \frac{\epsilon_0}{2} \int_{Q_s} \left(\frac{y_t^2}{\sigma} + \eta y_x^2 - \lambda \frac{y^2}{\sigma d} \right) dx dt \\ & \leq \left(2\Theta + \frac{1}{\sigma(1)} + \frac{1}{\eta(1)} + \frac{\beta}{\eta(1)} + \frac{K_a}{4} \right) E_y(s) \\ & \quad + \left(\frac{\beta^2}{\eta(1)} + \frac{K_a \beta}{2} + \frac{\beta}{\eta(1)} + \frac{K_a}{4} + \frac{\lambda}{\sigma(1)d(1)} \right) \int_s^T y^2(t, 1) dt, \end{aligned}$$

as claimed.

Case $\lambda \leq 0$. In this case, by definition of energy, one has

$$-\lambda \int_{Q_s} \frac{y^2}{\sigma d} dx dt \leq -2\lambda C_{HP} \int_s^T E_y(t) dt,^2$$

thus, by (3.53), (3.34), and by² estimating the boundary terms as in the previous case (with the exception that in (3.57) the term with λ disappears), we find

$$\begin{aligned} \frac{\varepsilon_0}{2} \int_{Q_s} \left(\frac{y_t^2}{\sigma} + \eta y_x^2 - \lambda \frac{y^2}{\sigma d} \right) dx dt &\leq \int_{Q_s} \left(1 - \frac{x(a' - b)}{a} + \frac{K_a}{2} \right) \frac{1}{\sigma} y_t^2 dx dt + \int_{Q_s} \left(1 - x \frac{b}{a} - \frac{K_a}{2} \right) \eta y_x^2 dx dt \\ &\leq \int_0^1 \left[\left(-2x \frac{y_x y_t}{\sigma} + \frac{K_a}{2} \frac{y y_t}{\sigma} \right) (\tau, x) dx \right]_{\tau=s}^{\tau=T} + \int_s^T \left[\frac{1}{\sigma} y_t^2 + \eta y_x^2 - \frac{K_a}{2} \eta y_x y + \frac{\lambda}{\sigma d} y^2 \right] (t, 1) dt \\ &\quad - \lambda \int_{Q_s} \left(1 - \frac{x(a' - b)}{a} - \frac{x d'}{d} + \frac{K_a}{2} \right) \frac{y^2}{\sigma d} dx dt \\ &\leq \left(2\Theta + \frac{1}{\sigma(1)} + \frac{1}{\eta(1)} + \frac{\beta}{\eta(1)} + \frac{K_a}{4} \right) E_y(s) + \left(\frac{\beta^2}{\eta(1)} + \frac{K_a \beta}{2} + \frac{\beta}{\eta(1)} + \frac{K_a}{4} \right) \int_s^T y^2(t, 1) dt \\ &\quad - 2\lambda C_{HP} \left(1 + \frac{3}{2} K_a + K_d + M \right) \int_s^T E_y(t) dt. \end{aligned}$$

Our proof is then concluded. $\square$

As a consequence of Propositions 5 and 6, we have the main result of the paper. As a first step, define

$$\begin{aligned} C_1 &:= 2\Theta + \frac{1}{\sigma(1)} + \frac{1}{\eta(1)} + \frac{\beta}{\eta(1)} + \frac{K_a}{4}, \\ C_2 &:= \begin{cases} \frac{\beta^2}{\eta(1)} + \frac{K_a \beta}{2} + \frac{\beta}{\eta(1)} + \frac{K_a}{4} + \frac{\beta \varepsilon_0}{2} + \frac{\lambda}{\sigma(1)d(1)}, & \lambda \geq 0, \\ \frac{\beta^2}{\eta(1)} + \frac{K_a \beta}{2} + \frac{\beta}{\eta(1)} + \frac{K_a}{4} + \frac{\beta \varepsilon_0}{2}, & \lambda < 0, \end{cases} \\ C_3 &:= \frac{2}{\mathbf{I}_\varepsilon} + \frac{2}{\mathbf{I}_\varepsilon^2 \min_{[0,1]}^2 \eta} \tilde{C}_{HP} C_\lambda^2 + \frac{1}{2\delta} + \frac{1}{2\delta} \left(\tilde{C}_{HP} + \max_{[0,1]} \eta \right) C_\lambda^4 \end{aligned}$$

and

$$C_4 := \frac{1}{\mathbf{I}_\varepsilon} \left(1 + \frac{1}{\mathbf{I}_\varepsilon \min_{[0,1]}^2 \eta} C_\lambda^2 \right)$$

where $\delta > 0$.

Theorem 3.2. Assume Hypothesis 4 and if $\lambda < 0$, then $\lambda \in \left(-\frac{\varepsilon_0}{2C_{HP}(1+\frac{3}{2}K_a+K_d+M)}, 0 \right)$. Let y be a mild solution of (1.1). Then, for all $t \geq \mathcal{M}$ and for all $\delta \in (0, \delta_0)$,

$$E_y(t) \leq E_y(0) e^{1-\frac{t}{\mathcal{M}}}, \quad (3.58)$$

²Of course, a straightforward estimate could be $-\lambda \int_{Q_s} \frac{y^2}{\sigma d} dx dt \leq 2 \int_s^T E_y(t) dt$, but we could not manage such a term in Theorem 3.2.

where $\delta_0 := \min \left\{ \frac{\varepsilon_0}{C_2 C_4}, \frac{\varepsilon_0 + 2\lambda_{HP} \left(1 + \frac{3}{2}K_a + K_d + M\right)}{C_2 C_4} \right\}$ and

$$\mathcal{M} := \begin{cases} \frac{C_1 + C_2 C_3}{\varepsilon_0 - C_2 C_4 \delta}, & \lambda \in \left[0, \frac{1}{C_{HP}}\right), \\ \frac{C_1 + C_2 C_3}{\varepsilon_0 - C_2 C_4 \delta + 2\lambda C_{HP} \left(1 + \frac{3}{2}K_a + K_d + M\right)}, & \lambda \in \left(-\frac{\varepsilon_0}{2C_{HP} \left(1 + \frac{3}{2}K_a + K_d + M\right)}, 0\right). \end{cases} \quad (3.59)$$

Proof. As usual, let us start assuming that y is a classical solution. If y is the mild solution associated to the initial data $(y_0, y_1) \in \mathcal{H}_0$, consider a sequence $\{(y_0^n, y_1^n)\}_{n \in \mathbb{N}} \in D(\mathcal{A})$ that approximate (y_0, y_1) and let y^n be the classical solution of (1.1) associated to (y_0^n, y_1^n) . With standard estimates coming from Theorem A.1, we can pass to the limit in (3.58) written for y^n and obtain the desired result. So, let y be a classical solution.

As a first step assume $\lambda \geq 0$. By (3.50) and (3.39), recalling the definition of E_y , we have

$$\begin{aligned} \varepsilon_0 \int_s^T E_y(t) dt &\leq C_1 E_y(s) + C_2 \int_s^T y^2(t, 1) dt \\ &\leq (C_1 + C_2 C_3) E_y(s) + C_2 C_4 \delta \int_s^T E_y(t) dt. \end{aligned}$$

Hence,

$$[\varepsilon_0 - C_2 C_4 \delta] \int_s^T E_y(t) dt \leq (C_1 + C_2 C_3) E_y(s).$$

Choosing $\delta \in \left(0, \frac{\varepsilon_0}{C_2 C_4}\right)$, we have

$$\int_s^T E_y(t) dt \leq \frac{C_1 + C_2 C_3}{\varepsilon_0 - C_2 C_4 \delta} E_y(s)$$

for all $s \in [0, +\infty)$. Thus, we can apply Fragnelli and Mugnai [38, lemma 3.1] obtaining

$$E_y(t) \leq E_y(0) e^{1 - \frac{t}{\mathcal{M}}},$$

for all $t \in [\mathcal{M}, +\infty)$, where $\mathcal{M}$ is as in the first line of (3.59).

If $\lambda < 0$, then from (3.51) and (3.39), we get

$$\begin{aligned} \varepsilon_0 \int_s^T E_y(t) dt &\leq C_1 E_y(s) + C_2 \int_s^T y^2(t, 1) dt - 2\lambda C_{HP} \left(1 + \frac{3}{2}K_a + K_d + M\right) \int_s^T E_y(t) dt \\ &\leq (C_1 + C_2 C_3) E_y(s) + C_2 C_4 \delta \int_s^T E_y(t) dt - 2\lambda C_{HP} \left(1 + \frac{3}{2}K_a + K_d + M\right) \int_s^T E_y(t) dt. \end{aligned}$$

Hence,

$$\left[\varepsilon_0 - C_2 C_4 \delta + 2\lambda C_{HP} \left(1 + \frac{3}{2}K_a + K_d + M\right) \right] \int_s^T E_y(t) dt \leq (C_1 + C_2 C_3) E_y(s).$$

Proceeding as before, with the choices of λ and δ in the statement of the theorem, one has the claim. $\square$

Remark 4. We notice that the condition on λ given in Theorem 3.2 is strictly related to the one given in Fragnelli et al. [36] for the null controllability result.

AUTHOR CONTRIBUTIONS

Genni Fragnelli: Investigation; conceptualization; methodology; validation; writing—original draft. **Dimitri Mugnai:** Conceptualization; investigation; methodology; validation; writing—original draft. **Amine Sbai:** Methodology; validation; investigation; writing—original draft.

ACKNOWLEDGEMENTS

G. Fragnelli is partially supported by INdAM GNAMPA Project 2023 “*Modelli differenziali per l'evoluzione del clima e i suoi impatti*” (CUP E53C22001930001) and by INdAM GNAMPA Project 2024 “*Analysis, control and inverse problems for evolution equations arising in climate science*” (CUP E53C23001670001). She is also a member of UMI “*Modellistica Socio-Epidemiologica (MSE)*”. D. Mugnai is partially supported by INdAM GNAMPA Project 2023 “*Variational and non-variational problems with lack of compactness*” (CUP E53C22001930001), by INdAM GNAMPA Project 2024 “*Nonlinear problems in local and nonlocal settings with applications*” (CUP E53C23001670001) and by the PRIN 2022 “*Advanced theoretical aspects in PDEs and their applications*” (CUP J53D23003700006). G. Fragnelli and D. Mugnai are members of UMI “*CliMath*” and of Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). This paper was partially written during the visit of A. Sbai at Università of Tuscia, under a grant of the MAECI (Ministry of Foreign Affairs and International Cooperation, Italy).

CONFLICTS OF INTEREST STATEMENT

This work does not have any conflicts of interest.

ORCID

Genni Fragnelli  <https://orcid.org/0000-0002-5436-7006>

Dimitri Mugnai  <https://orcid.org/0000-0001-8908-5220>

Amine Sbai  <https://orcid.org/0000-0003-0639-4584>

REFERENCES

1. W. Feller, *The parabolic differential equations and the associated semigroups of transformations*, Ann. Math. **55** (1952), 468–519.
2. G. Fragnelli and D. Mugnai, *Control of Degenerate and Singular Parabolic Equations. Carleman Estimates and Observability*, Springer International Publishing, SpringerBriefs in Mathematics, 2021.
3. N. I. Karachalios and N. B. Zographopoulos, *On the dynamics of a degenerate parabolic equation: global bifurcation of stationary states and convergence*, Calc. Var. Partial Differ. Equ. **25** (2006), 361–393.
4. M. Bertsch, M. E. Gurtin, and D. Hilhorst, *On a degenerate diffusion equation of the form $c(z)t = \phi(z_x)_x$ with application to population dynamic*, J. Differ. Equ. **67** (1987).
5. C. L. Epstein and R. Mazzeo, *Degenerate diffusion operators arising in population biology*. Annals of Mathematics Studies, Princeton University Press, 2013.
6. P. Hagan and D. Woodward, *Equivalent black volatilities*, Appl. Math. Finance **6** (1999), 147–159.
7. J.-I. Diaz, *Mathematical analysis of some diffusive energy balance models in climatology*, Math. Clim. Environ. RMA Res. Notes Appl. Math. Masson, Paris **27** (1993), 28–56.
8. G. Citti and M. Manfredini, *A degenerate parabolic equation arising in image processing*, Commun. Appl. Anal. **8** (2004), 125–141.
9. F. Alabau-Boussouira, P. Cannarsa, and G. Fragnelli, *Carleman estimates for degenerate parabolic operators with applications to null controllability*, J. Evol. Equ. **6** (2006), 161–204.
10. A. Benoit, R. Loyer, and L. Rosier, *Null controllability of strongly degenerate parabolic equations*, ESAIM Control Optim. Calc. Var. **29** (2023), no. 48, 35 pp.
11. I. Boutaayamou, G. Fragnelli, and L. Maniar, *Carleman estimates for parabolic equations with interior degeneracy and Neumann boundary conditions*, J. Anal. Math. **135** (2018), 1–35.
12. P. Cannarsa, G. Fragnelli, and D. Rocchetti, *Controllability results for a class of one-dimensional degenerate parabolic problems in nondivergence form*, J. Evol. Equ. **8** (2008), 583–616.

13. P. Cannarsa, G. Fragnelli, and D. Rocchetti, *Null controllability of degenerate parabolic operators with drift*, *Netw. Heterog. Media* **2** (2007), 693–713.
14. M. M. Cavalcanti, E. Fernández-Cara, and A. L. Ferreira, *Null controllability of some nonlinear degenerate 1D parabolic equations*, *J. Franklin Inst.* **354** (2017), 6405–6421.
15. G. Fragnelli and D. Mugnai, *Carleman estimates and observability inequalities for parabolic equations with interior degeneracy*, *Adv. Nonlin. Anal.* **2** (2013), 339–378.
16. G. Fragnelli and D. Mugnai, *Carleman estimates, observability inequalities and null controllability for interior degenerate nonsmooth parabolic equations*, *Mem. Amer. Math. Soc.* **242** (2016), no. 1146, 96 pp.
17. H. Gao, W. Yang, and M. Zhang, *Hierarchical control for the semilinear parabolic equations with interior degeneracy*, *J. Math. Anal. Appl.* **534** (2024), 128092.
18. C. Wang and Y. Zhou, *Carleman estimate and null controllability for a degenerate parabolic equation with a slightly superlinear reaction term*, *NoDEA Nonlin. Differ. Equ. Appl.* **30** (2023), no. 69.
19. B. Allal, G. Fragnelli, and J. Salhi, *On a general degenerate/singular parabolic equation with a nonlocal space term*, *Math. Methods Appl. Sci.* **46** (2023), 12473–12504.
20. B. Allal, A. Hajjaj, J. Salhi, and A. Sbair, *Boundary controllability for a coupled system of degenerate/singular parabolic equations*, *Evol. Equ. Control Theory* **11** (2022), 1579–1604.
21. C. T. Anh, V. M. Toi, and T. Q. Tuan, *Lipschitz stability in inverse source problems for degenerate/singular parabolic equations by the Carleman estimate*, *J. Inverse Ill-Posed Probl.* **31** (2023), 313–325.
22. U. Biccari, V. Hernández-Santamaría, and J. Vancostenoble, *Existence and cost of boundary controls for a degenerate/singular parabolic equation*, *Math. Control Relat. Fields* **12** (2022), 495–530.
23. G. Fragnelli, *Interior degenerate/singular parabolic equations in nondivergence form: well-posedness and Carleman estimates*, *J. Differ. Equ.* **260** (2016), 1314–1371.
24. G. Fragnelli and D. Mugnai, *Carleman estimates for singular parabolic equations with interior degeneracy and non-smooth coefficients*, *Adv. Nonlin. Anal.* **6** (2017), 61–84.
25. G. Fragnelli and D. Mugnai, *Controllability of strongly degenerate parabolic problems with strongly singular potentials*, *Electron. J. Qual. Theory Differ. Equ.* **2018** (2018), no. 50, 1–11.
26. M. Akil, G. Fragnelli, and I. Issa, *Stability for degenerate wave equations with drift under simultaneous degenerate damping*, *J. Differ. Equ.* **416** (2025), 1178–1221 submitted preprint; arXiv:2308.08645v3.
27. F. Alabau-Boussouira, P. Cannarsa, and G. Leugering, *Control and stabilization of degenerate wave equations*, *SIAM J. Control Optim.* **55** (2017), 2052–2087.
28. B. S. V. Araújo, R. Demarque, and L. Viana, *Regularity results for degenerate wave equations in a neighborhood of the boundary*, *Evol. Equ. Control Theory* **12** (2023), 1300–1315.
29. J. Bai, S. Chai, and Z. Guo, *Indirect internal controllability of weakly coupled degenerate wave equations*, *Acta Appl. Math.* **180** (2022), no. 7.
30. I. Boutaayamou, G. Fragnelli, and D. Mugnai, *Boundary controllability for a degenerate wave equation in non divergence form with drift*, *SIAM J. Control Optim.* **61** (2023), 1934–1954.
31. Y. Liu and W. Xie, *Exact controllability of degenerate wave equations with locally distributed control in moving boundary domain*, *Acta Appl. Math.* **177** (2022), no. 11.
32. G. Zhang and S. Chai, *Stabilization for some degenerate wave equations*, *Evol. Equ. Control Theory* **30** (2024), 316–328.
33. C. Cazacu, *Controllability of the heat equation with an inverse-square potential localized on the boundary*, *SIAM J. Control Optim.* **52** (2014), 2055–2089.
34. J. Vancostenoble and E. Zuazua, *Hardy inequalities, observability, and control for the wave and Schrödinger equations with singular potentials*, *SIAM Journal on Mathematical Analysis* **41** (2009), 1508–1532.
35. B. Allal, A. Moumni, and J. Salhi, *Boundary controllability for a degenerate and singular wave equation*, *Math. Meth. Appl. Sci.* **45** (2022), 11526–11544.
36. G. Fragnelli, D. Mugnai, and A. Sbair, *Boundary controllability for degenerate/singular hyperbolic equations with drift*. arXiv:2402.18247v1 submitted.
37. A. Moumni and J. Salhi, *Exact controllability for a degenerate and singular wave equation with moving boundary*, *Numer. Algebra Control Optim.* **13** (2023), 194–209.
38. G. Fragnelli and D. Mugnai, *Linear stabilization for a degenerate wave equation in non divergence form with drift*. *Adv. Differential Equations to appear*. arXiv:2212.05264v1.
39. K. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, GTM 194: Springer, New York, 2000.
40. A. Bensoussan, G. D. Prato, M. C. Delfour, and S. K. Mitter, *A Representation and Control of Infinite Dimensional Systems*, Second Edition, Birkhauser Boston, Inc., Boston, MA, 2007.

How to cite this article: G. Fragnelli, D. Mugnai, and A. Sbair, *Stabilization for degenerate equations with drift and small singular term*, *Math. Meth. Appl. Sci.* **48** (2025), 5086–5109, DOI 10.1002/mma.10593.

APPENDIX A

For the readers' convenience, in this section, we prove some results used throughout the previous sections.

A.1 | Proof of Theorem 2.1

In order to prove Theorem 2.1, we introduce the Hilbert space

$$\mathcal{H}_0 := H_{\frac{1}{\sigma},0}^1(0,1) \times L_{\frac{1}{\sigma}}^2(0,1), \quad (\text{A1})$$

endowed with the inner product

$$\langle (u, v), (\tilde{u}, \tilde{v}) \rangle_{\mathcal{H}_0} := \int_0^1 u' \tilde{u}' dx + \int_0^1 v \tilde{v} \frac{1}{\sigma} dx - \lambda \int_0^1 \frac{u \tilde{u}}{\sigma d} dx + \beta u(1) \tilde{u}(1)$$

for every $(u, v), (\tilde{u}, \tilde{v}) \in \mathcal{H}_0$, which induces the norm

$$\|(u, v)\|_{\mathcal{H}_0}^2 := \int_0^1 (u')^2 dx + \int_0^1 v^2 \frac{1}{\sigma} dx - \lambda \int_0^1 \frac{u^2}{\sigma d} dx + \beta u^2(1).$$

Observe that if $u \in H_{\frac{1}{\sigma},0}^1(0,1)$, then u is continuous, so that $u(1)$ is well defined. Moreover, being $\eta \in C^0[0,1] \cap C^1(0,1]$ and bounded away from 0, the norm $\|(u, v)\|_{\mathcal{H}_0}^2$ is equivalent to

$$\|(u, v)\|_1^2 := \int_0^1 \eta (u')^2 dx + \int_0^1 v^2 \frac{1}{\sigma} dx - \lambda \int_0^1 \frac{u^2}{\sigma d} dx + \beta u^2(1), \quad (u, v) \in \mathcal{H}_0.$$

Obviously, to such a norm, we associate the inner product

$$\langle (u, v), (\tilde{u}, \tilde{v}) \rangle_1 := \int_0^1 \eta u' \tilde{u}' dx + \int_0^1 v \tilde{v} \frac{1}{\sigma} dx - \lambda \int_0^1 \frac{u \tilde{u}}{\sigma d} dx + \beta u(1) \tilde{u}(1),$$

which we will use from now on, being more convenient for our treatment.

In order to study the well posedness of (1.1), we introduce the matrix operator $(\mathcal{A}, D(\mathcal{A}))$ given by

$$\mathcal{A} := \begin{pmatrix} 0 & Id \\ A_\lambda & 0 \end{pmatrix},$$

$$D(\mathcal{A}) := \left\{ (u, v) \in D(A_\lambda) \times H_{\frac{1}{\sigma},0}^1(0,1) : \eta u'(1) + v(1) + \beta u(1) = 0 \right\} \subset \mathcal{H}_0,$$

where $(A_\lambda, D(A_\lambda))$ is defined in (2.17). In this way, we can rewrite (1.1) as the Cauchy problem

$$\begin{cases} \dot{\mathcal{Y}}(t) = \mathcal{A}\mathcal{Y}(t), & t \geq 0, \\ \mathcal{Y}(0) = \mathcal{Y}_0, \end{cases} \quad (\text{A2})$$

with

$$\mathcal{Y}(t) := \begin{pmatrix} y \\ y_t \end{pmatrix} \quad \text{and} \quad \mathcal{Y}_0 := \begin{pmatrix} y_T^0 \\ y_T^1 \end{pmatrix}.$$

Theorem A.1. Assume Hypotheses 1 and 2 and let $K_a + 2K_d \leq 2$. Then the operator $(\mathcal{A}, D(\mathcal{A}))$ is nonpositive with dense domain and generates a contraction semigroup $(S(t))_{t \geq 0}$.

Proof. The proof is based on Engel and Nagel [39, Corollary 3.20]. According to this result, it is sufficient to prove that $\mathcal{A} : D(\mathcal{A}) \rightarrow \mathcal{H}_0$ is dissipative and $I - \mathcal{A}$ is surjective.

$\mathcal{A}$ is dissipative: take $(u, v) \in D(\mathcal{A})$. Then $(u, v) \in H_{\frac{1}{\sigma},0}^2(0,1) \times H_{\frac{1}{\sigma},0}^1(0,1)$, and so Lemma 2 holds. Hence,

$$\begin{aligned} \langle \mathcal{A}(u, v), (u, v) \rangle_{\mathcal{H}_0} &= \langle (v, A_\lambda u), (u, v) \rangle_{\mathcal{H}_0} \\ &= \int_0^1 \eta u' v' dx + \int_0^1 v A_\lambda u \frac{1}{\sigma} dx - \lambda \int_0^1 \frac{uv}{\sigma d} dx + \beta u(1)v(1) \\ &= \int_0^1 \eta u' v' dx - \int_0^1 \eta u' v' dx + [\eta u' v](1) + \lambda \int_0^1 \frac{uv}{\sigma d} dx - \lambda \int_0^1 \frac{uv}{\sigma d} dx \\ &\quad + \beta u(1)v(1) = -v^2(1) \leq 0. \end{aligned}$$

$I - \mathcal{A}$ is surjective: take $(f, g) \in \mathcal{H}_0 = H_{\frac{1}{\sigma},0}^1(0,1) \times L_{\frac{1}{\sigma}}^2(0,1)$. We have to prove that there exists $(u, v) \in D(\mathcal{A})$ such that

$$(I - \mathcal{A}) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \iff \begin{cases} v = u - f, \\ -A_\lambda u + u = f + g. \end{cases} \quad (\text{A3})$$

Thus, define $F : H_{\frac{1}{\sigma},0}^1(0,1) \rightarrow \mathbb{R}$ as

$$F(z) = \int_0^1 (f + g)z \frac{1}{\sigma} dx + z(1)f(1).$$

Obviously, $F \in H_{\frac{1}{\sigma},0}^{-1}(0,1)$, the dual space of $H_{\frac{1}{\sigma},0}^1(0,1)$ with respect to the pivot space $L_{\frac{1}{\sigma}}^2(0,1)$. Now, introduce the bilinear form $L : H_{\frac{1}{\sigma},0}^1(0,1) \times H_{\frac{1}{\sigma},0}^1(0,1) \rightarrow \mathbb{R}$ given by

$$L(u, z) := \int_0^1 uz \frac{1}{\sigma} dx + \int_0^1 \eta u' z' dx - \lambda \int_0^1 \frac{uz}{\sigma d} dx + (\beta + 1)u(1)z(1)$$

for all $u, z \in H_{\frac{1}{\sigma},0}^1(0,1)$. Clearly, $L(u, z)$ is coercive: indeed, for all $u \in H_{\frac{1}{\sigma},0}^1(0,1)$, we have

$$\begin{aligned} L(u, u) &= \int_0^1 \frac{u^2}{\sigma} dx + \int_0^1 \eta (u')^2 dx - \lambda \int_0^1 \frac{u^2}{\sigma d} dx + (\beta + 1)u^2(1) \\ &\geq \int_0^1 \frac{u^2}{\sigma} dx + \int_0^1 \eta (u')^2 dx = \|u\|_{1, \frac{1}{\sigma}}^2, \end{aligned}$$

if $\lambda \leq 0$. On the other hand, if $\lambda \in (0, C_{HP})$ then

$$L(u, u) \geq \int_0^1 \frac{u^2}{\sigma} dx + (1 - \lambda C_{HP}) \int_0^1 \eta (u')^2 dx \geq (1 - \lambda C_{HP}) \|u\|_{1, \frac{1}{\sigma}}^2,$$

by (2.13). Moreover, $L(u, z)$ is continuous: indeed, using again (2.13), one has

$$\begin{aligned} |L(u, z)| &\leq \|u\|_{L_{\frac{1}{\sigma}}^2(0,1)} \|z\|_{L_{\frac{1}{\sigma}}^2(0,1)} \\ &\quad + (\|\eta\|_{L^\infty(0,1)} + |\lambda| \|\eta\|_{L^\infty(0,1)} C_{HP} + (\beta + 1)) \|u'\|_{L^2(0,1)} \|z'\|_{L^2(0,1)} \end{aligned}$$

for all $u, z \in H_{\frac{1}{\sigma},0}^1(0,1)$, being

$$|u(1)| \leq \int_0^1 |u'(x)| dx \leq \|u'\|_{L^2(0,1)}$$

(analogously for $z(1)$), and the conclusion follows.

As a consequence, by the Lax–Milgram The, there exists a unique solution $u \in H_{\frac{1}{\sigma},0}^1(0,1)$ of

$$L(u, z) = F(z) \text{ for all } z \in H_{\frac{1}{\sigma},0}^1(0,1),$$

namely

$$\int_0^1 \frac{uz}{\sigma} dx + \int_0^1 \eta u' z' dx - \lambda \int_0^1 \frac{uz}{\sigma d} dx + (\beta + 1)u(1)z(1) = \int_0^1 (f + g)z \frac{1}{\sigma} dx + z(1)f(1) \quad (\text{A4})$$

for all $z \in H_{\frac{1}{\sigma},0}^1(0,1)$.

Now, take $v := u - f$; then $v \in H_{\frac{1}{\sigma},0}^1(0,1)$. We will prove that $(u, v) \in D(\mathcal{A})$ and solves (A3). To begin with, (A4) holds for every $z \in C_c^\infty(0,1)$. Thus, we have

$$\int_0^1 \eta u' z' dx = \int_0^1 \left(f + g - u + \lambda \frac{u}{d} \right) z \frac{1}{\sigma} dx$$

for every $z \in C_c^\infty(0,1)$. Hence, $-(\eta u')' = \left(f + g - u + \lambda \frac{u}{d} \right) \frac{1}{\sigma}$ a.e. in $(0,1)$, i.e. $-\sigma(\eta u')' = \left(f + g - u + \lambda \frac{u}{d} \right)$ a.e. in $(0,1)$. In particular, $A_\lambda u = Au + \lambda \frac{u}{d} = -(f + g - u) \in L_{\frac{1}{\sigma}}^2(0,1)$. Moreover, using the assumption $K_a + 2K_d \leq 2$, one has that $\frac{u}{d} \in L_{\frac{1}{\sigma}}^2(0,1)$; thus $u \in D(A_\lambda)$. Coming back to (A4) and thanks to Lemma 2, one has

$$-\int_0^1 \sigma(\eta u')' z \frac{1}{\sigma} dx + (\eta u' z)(1) + (\beta + 1)u(1)z(1) = \int_0^1 \left(f + g - u + \lambda \frac{u}{d} \right) z \frac{1}{\sigma} dx + z(1)f(1)$$

and

$$(\eta u' z)(1) + (\beta + 1)u(1)z(1) = z(1)f(1)$$

for all $z \in H_{\frac{1}{\sigma},0}^1(0,1)$, since $-\sigma(\eta u')' = \left(f + g - u + \lambda \frac{u}{d} \right)$ a.e. in $(0,1)$. Hence, $\eta(1)u'(1) + (\beta + 1)u(1) - f(1) = 0$. Recalling that $v = u - f$, one has

$$\eta(1)u'(1) + \beta u(1) + v(1) = 0.$$

In conclusion, $(u, v) \in D(\mathcal{A})$ and (A3) holds. $\square$

As usual in semigroup theory, the mild solution of (A2) obtained above can be more regular: if $\mathcal{Y}_0 \in D(\mathcal{A})$, then the solution is classical, in the sense that $\mathcal{Y} \in C^1([0, +\infty); H_0) \cap C([0, +\infty); D(\mathcal{A}))$ and the equation in (1.1) holds for all $t \geq 0$. Hence, as in Alabau-Boussouira et al. [27, corollary 4.2] or in Bensoussan et al. [40, proposition 3.15], one can deduce Theorem 2.1.