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RESEARCH ARTICLE

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Fractional weighted problems with a general nonlinearity or with concave-convex nonlinearities

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We consider nonlocal problems in which the leading operator contains a sign-changing weight which can be unbounded. We begin studying the existence and the properties of the first eigenvalue. Then we study a nonlinear problem in which the nonlinearity does not satisfy the usual Ambrosetti-Rabinowitz condition. Finally, we study a problem with general concave-convex nonlinearities.

KEYWORDS

convex and concave nonlinearities, first eigenvalue, fractional Laplacian, indefinite weight, superlinear problems

MSC CLASSIFICATION

35R11; 35B38; 45C05; 45E10

1 | INTRODUCTION

We are concerned with a class of nonlinear nonlocal problems in presence of a weight β, possibly unbounded, which is allowed to change sign. The prototype equations are

$$\begin{cases} (-\Delta)^s u + \beta(x)u = h(\lambda, x, u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

but, actually, we shall consider problems where the leading operator $(-\Delta)^s$ is replaced by more general nonlocal ones denoted by $\mathfrak{L}_K$, see Section 2 for the precise setting. Here $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $\lambda \in \mathbb{R}$ and h satisfies suitable structure conditions.

We shall start analyzing the eigenvalue problem

$$\begin{cases} \mathfrak{L}_K u + \beta(x)u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1}$$

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showing the existence of a principal eigenvalue $\hat{\lambda}_1$ enjoying the usual properties of the first eigenvalue in the classical locale case. This fact is far from being trivial, due to the fact that, at this step, β is assumed to be unbounded and sign–changing. Once the existence of $\hat{\lambda}_1$ is proved, it is standard to show the existence of a diverging sequence of eigenvalues solving (1), see Theorem 2 below.

After this preliminary result, we will look for solutions to problems of the form

$$\begin{cases} \mathfrak{L}_K u + \beta(x)u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2}$$

with different assumptions on f. In particular, we produce two constant sign solutions in Theorem 3 by using the Mountain Pass Theorem.

Finally, we consider a problem of the form

$$\begin{cases} \mathfrak{L}_K u + \beta(x)u = \lambda g(x,u) + f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3}$$

where $g(x,\cdot)$ has sublinear growth at infinity, while $f(x,\cdot)$ exhibits a superlinear growth. In this case we find two constant sign solutions, and we produce a third nontrivial one by using the Weiestrass Theorem, provided that λ is positive and small. Of course, this result has the flavor of the celebrated one in previous work[1] for the local case. However, we shall treat a nonlinear source f which does not satisfy the usual Ambrosetti–Rabinowitz condition (AR-condition for short), as done in previous work[2] for the local Neumann case. Indeed, we employ a more general condition introduced in previous work,[3] which covers the case of superlinear reactions with slower growth near $\pm\infty$ and which fail to satisfy the AR-condition; of course, the lack of the AR-condition makes the situation more complicated, since it is not clear if Palais-smale sequences are bounded. Thus, our result improves those in previous work,[4] where the existence of two solutions when $\beta = 0$ is proved in presence of pure powers. For related results, see also previous work[5] for the spectral fractional Laplacian, recalling that such an operator is quite different from the one considered here, see previous work[6, Section 2.3] for a detailed discussion on this fact. We also mention,[7] where a probem like (3) with pure powers and with f having critical growth has been studied in presence of continuous and sign changing coefficients, showing the existence of two positive solutions for λ small enough. We conclude recalling that many other concave–convex problems have been studied in different situations, for instance, in previous works.[8–11]

2 | MATHEMATICAL BACKGROUND

The underlying operator $\mathfrak{L}_K$ is defined, up to a positive multiplicative constant which we rescale to 1, as

$$\mathfrak{L}_K u(x) = 2PV \int_{\mathbb{R}^N} (u(x) - u(y))\, K(x-y)\ dy,$$

PV denoting the Cauchy principal value, namely,

$$PV \int_{\mathbb{R}^N} (u(x) - u(y))\, K(x-y)\ dy = \lim_{\epsilon \to 0} \int_{\{y \in \mathbb{R}^N \ :\ |y| \geq \epsilon\}} (u(x) - u(y))\, K(x-y)\ dy.$$

Moreover, $K : \mathbb{R}^N \setminus \{0\} \to (0, \infty)$ is a function satisfying the following

***κ*-assumption**:

1. $\gamma K \in L^1(\mathbb{R}^N)$, where $\gamma(x) = \min\left\{1, |x|^2\right\}$;
2. there exist $\varkappa > 0$ such that $K(x) \geq \varkappa\, |x|^{-(N+2s)}$ for any $x \in \mathbb{R}^N \setminus \{0\}$.

We notice that $\mathfrak{L}_K = (-\Delta)^s$ when $K(x) = |x|^{-(N+2s)}$.

Remark 1. Usually, it is also assumed that $K(x) = K(-x)$ for any $x \in \mathbb{R}^N \setminus \{0\}$. However, as shown in previous works,[12,13] one can equally prove that

$$\mathfrak{L}_K u(x) = \int_{\mathbb{R}^N} [2u(x) - u(x+y) - u(x-y)]\, K(y) dy$$

without such a condition, thus we will not assume it.

In order to work with the operator $\mathfrak{L}_K$, it is necessary to introduce a suitable functional setting.
From now on, we fix $s \in (0,1)$, $N > 2s$, and $\Omega \subset \mathbb{R}^N$ an open bounded set with Lipschitz Boundary. The space X is

$$X = \left\{ v : \mathbb{R}^N \to \mathbb{R} : v|_\Omega \in L^2(\Omega),\ (v(x) - v(y)) \sqrt{K(x-y)} \in L^2(Q) \right\},$$

where $Q = \mathbb{R}^{2N} \setminus \mathcal{O}$ and $\mathcal{O} = \Omega^c \times \Omega^c$. The space X is endowed with the norm

$$\|v\|_X = \|v\|_{L^2(\Omega)} + \left(\int_Q |v(x) - v(y)|^2 K(x-y)\ dxdy \right)^{\frac{1}{2}}.$$

Moreover, we set

$$X_0 = \left\{ v \in X :\ \ v = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}.$$

Like in the case of Sobolev spaces with integer s, it is possible to define a critical exponent that plays the same role in the embedding theorems. Precisely we define

$$2^\star = \frac{2N}{N - 2s},$$

and we have the following

Lemma 1 ([14, Lemma 6]). *Let $K : \mathbb{R}^N \setminus \{0\} \to (0, \infty)$ satisfy the **κ-assumption**. Then*

1. *there exists a positive constant $c = c(N, s)$, such that for any $v \in X_0$*

$$\|v\|^2_{L^{2^\star}(\Omega)} = \|v\|^2_{L^{2^\star}(\mathbb{R}^N)} \le c \int_Q \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s}}\ dxdy;$$

2. *there exist a constant $C = (N, s, \lambda, \Omega) > 1$ such that for any $v \in X_0$*

$$\int_Q |v(x) - v(y)|^2 K(x-y)\ dxdy \le \|v\|_X^2 \le C \int_Q |v(x) - v(y)|^2 K(x-y)\ dxdy,$$

that is

$$\|v\|_{X_0} = \left(\int_Q |v(x) - v(y)|^2 K(x-y)\ dxdy \right)^{\frac{1}{2}}$$

is a norm in X_0 equivalent to the usual one defined in X.

Lemma 2. ([14, Lemma 7]). $(X_0, \|\cdot\|_{X_0})$ *endowed with the scalar product*

$$\langle u, v \rangle_{X_0} = \int_Q (u(x) - u(y))\, (v(x) - v(y))\, K(x-y)\ dxdy$$

is a Hilbert space.

Recalling that Ω has a Lipschitz boundary, we have:

Lemma 3 ([14, Lemma 6 and 8]). *Let $K : \mathbb{R}^N \setminus \{0\} \to (0, \infty)$ satisfies the **κ-assumption**. Then the following assertions hold true:*

1. *the embedding $X_0 \hookrightarrow L^p(\mathbb{R}^N)$ is compact for every $p \in [1, 2^\star)$;*
2. *the embedding $X_0 \hookrightarrow L^{2^\star}(\mathbb{R}^N)$ is continuous.*

A fundamental compactness tool is the following

Definition 1. Let X be a Banach Space, and let X^* be its topological dual. Let $\varphi \in C^1(X)$; we say that φ satisfies the Cerami condition—(C) for short—if the following holds: every sequence $(u_n)_n \subset X$ such that

$$(\varphi(u_n))_n \subset \mathbb{R} \text{ is bounded and } (1 + \|u_n\|_X)\varphi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence.

We shall use the following variant of the Mountain Pass Theorem, where the original Palais-Smale condition is replaced by (C), see[15] for a proof.

Theorem 1 (Mountain Pass Theorem). *If X is a Banach space, $\varphi \in C^1(X)$ satisfies (C), u_0, u_1 satisfy $\|u_1 - u_0\|_X > \rho > 0$*

$$\max\{\varphi(u_0), \varphi(u_1)\} \le \inf\{\varphi(u) \ : \ \|u - u_0\|_X = \rho\} = \eta_\rho,$$

set $\Gamma := \{\gamma \in C([0,1], X) \ \ \gamma(0) = u_0, \ \gamma(1) = u_1\}$ and

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \varphi(\gamma(t)),$$

then $c \ge \eta_\rho$ and c is a critical value for φ.

3 | THE EIGENVALUE PROBLEM

In this section we give some results about the following nonlocal eigenvalue problem:

$$\begin{cases} \mathfrak{L}_K u + \beta(x)u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{P_λ}$$

where $\mathfrak{L}_K$ and Ω are as above. More precisely, we prove

Theorem 2. *Let K satisfy the **κ-assumption** and let $\beta \in L^q(\Omega)$ with $q > \frac{2^*}{2^*-2} = \frac{N}{2s}$. Then there exists a diverging sequence $(\hat\lambda_n)_n$ and associated eigenfunctions $(\hat u_n)_n \subset X_0 \setminus \{0\}$ such that $(\hat\lambda_n, \hat u_n)$ solve $(P_{\hat\lambda_n})$ for any $n \in \mathbb{N}$. Moreover, $\hat\lambda_1$ is simple with associated eigenfunction $\hat u_1 \ge 0$ a.e. in Ω.*

The proof of Theorem 2 essentially consists in proving that the candidate first eigenvalue is finite, and this is the hardest part, because β is unbounded and sign-changing. Once the finiteness of $\hat\lambda_1$ is proved, the existence of a diverging sequence of eigenvalues follows in a standard way by applying the classical genus theory to a perturbed functional. Hence, we start from

Proposition 1. *Let K satisfy the **κ-assumption** and let $\beta \in L^q(\Omega)$ with $q > \frac{N}{2s}$. Then problem (P_λ) has a smallest eigenvalue $\hat\lambda_1 \in \mathbb{R}$ which is simple and has an eigenfunction $\hat u_1 \in X_0$ such that $\hat u_1 \ge 0$ a.e. in Ω.*

Remark 2. If $\beta^+ \in L^\infty_{\text{loc}}(\Omega)$, or $K(x) = |x|^{-(N+2s)}$, we can conclude that $\hat u_1 > 0$ in Ω, for instance, see Jaros and Weth[16] or Del Pezzo and Quaas[17, Remark 1.3]

Proof of Proposition 1. Let $\Psi : X_0 \to \mathbb{R}$ be the functional defined by

$$\Psi(u) = \|u\|_{X_0}^2 + \int_\Omega \beta u^2 \, dx$$

and consider the set

$$M = \left\{ u \in X_0 : \int_\Omega u^2 dx = 1 \right\}.$$

Set

$$\hat{\lambda}_1 = \inf_{u \in M} \Psi(u). \tag{4}$$

Claim 1: $\hat{\lambda}_1 > -\infty$.

Note that $q > \frac{N}{2s}$, hence $2q' < 2^\star$. Then, if $u \in X_0$, by Theorem 3 we have that $u^2 \in L^{q'}(\Omega)$. Hence, by Hölder's inequality, we have that

$$\left| \int_\Omega \beta u^2 \, dz \right| \leq \|\beta\|_q \|u\|_{2q'}^2. \tag{5}$$

We know that $X_0 \hookrightarrow L^{2q'}(\Omega) \hookrightarrow L^2(\Omega)$ and the first embedding is compact. So, by Ehrling's inequality (for instance, see,[18, Lemma 10.1.28] given $\epsilon > 0$ we can find $c(\epsilon) > 0$ such that

$$\|u\|_{2q'}^2 \leq \epsilon \|u\|_{X_0}^2 + c(\epsilon)\|u\|_2^2 \; \forall \; u \in X_0. \tag{6}$$

From (5) and (6) we get

$$\|u\|_{X_0}^2 - \int_\Omega \beta u^2 \, dz \leq \|u\|_{X_0}^2 + \epsilon \|\beta\|_q \|u\|_{X_0}^2 + c(\epsilon)\|\beta\|_q \|u\|_2^2. \tag{7}$$

Now, we choose $\epsilon \in (0, 1/\|\beta\|_q)$. Reordering the terms from (7), we have

$$0 \leq \|u\|_{X_0}^2 (1 - \epsilon \|\beta\|_q) \leq \Psi(u) + c(\epsilon)\|\beta\|_q \|u\|_2^2, \tag{8}$$

hence

$$-c(\epsilon)\|\beta\|_q \|u\|_2^2 \leq \Psi(u),$$

which implies $\hat{\lambda}_1 > -\infty$.

Claim 2: The infimum is obtained at a function $\hat{u}_1 \in M$ with $\hat{u}_1 \geq 0$ in Ω.

Let $(u_n)_n \subset M$ be a minimizing sequence for (4), i.e. $\Psi(u_n) \to \hat{\lambda}_1$ as $n \to \infty$. Now, from (8) we observe that $(u_n)_n$ is bounded, so we may assume that

$$u_n \rightharpoonup \hat{u}_1 \text{ in } X_0 \text{ and } u_n \to \hat{u}_1 \text{ in } L^{2q'}(\Omega) \text{ as } n \to \infty.$$

By the weak sequential lower semicontinuity and Lemma 3, we have that

$$\|\hat{u}_1\|_{X_0}^2 \leq \liminf_{n\to\infty} \|u_n\|_{X_0}^2 \text{ and } \lim_{n\to\infty} \int_\Omega \beta u_n^2 \, dx = \int_\Omega \beta \hat{u}_1^2 \, dx,$$

and thus, $\Psi(\hat{u}_1) \leq \hat{\lambda}_1$. Since $\hat{u}_1 \in M$, we get that $\Psi(\hat{u}_1) = \hat{\lambda}_1$.

By the Lagrange multiplier rule, we have that $\hat{u}_1$ solves problem $(P_{\hat{\lambda}_1})$, and so $\hat{u}_1 \in X_0$ is an associated eigenfunction to $\hat{\lambda}_1$.

We observe that if u is a normalized eigenfunction for $(P_{\hat{\lambda}_1})$, by the triangle inequality we have

$$\begin{aligned}\hat{\lambda}_1 \le \Psi(|u|) &= \iint_D (|u(x)| - |u(y)|)^2 K(x-y)\ dxdy + \int_\Omega \beta u^2\ dx \\ &\le \iint_D (u(x) - u(y))^2 K(x-y)\ dxdy + \int_\Omega \beta u^2\ dx = \Psi(u) = \hat{\lambda}_1,\end{aligned}$$

hence we may assume $u \ge 0$.

Claim 3: $\hat{\lambda}_1$ is simple.

We start noticing that

$$\langle u^+, u^- \rangle_{X_0} = -\int_Q \left[u^+(y)u^-(x) + u^+(x)u^-(y)\right] K(x-y)\ dxdy \le 0 \tag{9}$$

for every $u \in X_0$.

Now we improve Claim 2, showing that any weak solution $u \in X_0$ of $(P_{\hat{\lambda}_1})$, $u \ne 0$, is such that either

$$u \ge 0 \text{ in } \Omega$$

or

$$u \le 0 \text{ in } \Omega.$$

Without loss of generality we assume that $\|u\|_2 = 1$ and by (9) we have

$$\begin{aligned}\hat{\lambda}_1 &= \Psi(u) = \Psi(u^+) + \Psi(u^-) - 2\langle u^+, u^- \rangle_{X_0} \\ &\ge \hat{\lambda}_1 \|u^+\|_2^2 + \hat{\lambda}_1 \|u^-\|_2^2 = \hat{\lambda}_1.\end{aligned}$$

Hence, in the previous inequality we find all equalities, and since $\Psi(u^\pm) \ge \hat{\lambda}_1 \|u^\pm\|_2^2$, we finally get the equalities

$$\Psi(u^+) = \hat{\lambda}_1 \|u^+\|_2^2 \text{ and } \Psi(u^-) = \hat{\lambda}_1 \|u^-\|_2^2,$$

that is u^+ and u^- are weak solution of $(P_{\hat{\lambda}_1})$, as well. Moreover, we also get that $\langle u^+, u^- \rangle_{X_0} = 0$, that is

$$0 = \int_Q \left[u^+(y)u^-(x) + u^+(x)u^-(y)\right] K(x-y)\ dxdy.$$

Since $K > 0$, we get that

$$u^+(y)u^-(x) + u^+(x)u^-(y) = 0 \text{ a.e. in } Q \text{ and so in } \Omega.$$

As a consequence, $u^- = 0$, or $u^+ = 0$, as claimed.

Finally, in order to prove that $\hat{\lambda}_1$ is simple, let us suppose that u, v are nontrivial solutions of $(P_{\hat{\lambda}_1})$. We have shown above that we can suppose $u, v \ge 0$ with $\int_\Omega u > 0$ and $\int_\Omega v > 0$. Hence, it is possible to solve the equation in α

$$0 = \int_\Omega (u - \alpha v)\ dx = \int_\Omega u\ dx - \alpha \int_\Omega v\ dx.$$

Recalling that $u - \alpha v$ is a solution of $(P_{\hat{\lambda}_1})$ as well, we have just seen that there are two available options: $u - \alpha v \ge 0$ with $u - \alpha v \ne 0$ or $u - \alpha v \equiv 0$; in the first case we would have $\int_\Omega (u - \alpha v)\ dz > 0$, which is a contradiction. Thus, we deduce that $u = \alpha v$, which proves the simplicity of $\hat{\lambda}_1$. □

Remark 3. If $\beta \in L^\infty(\Omega)$, then $\hat{u}_1 \in L^\infty(\Omega)$ by Iannizzotto et al,[19] and so by Ros-Oton and Serra[20, Prop. 1.1] we get that $u \in C^s(\bar{\Omega})$.

Remark 4. From now on we will denote by $\hat{u}_1$ the first eigenfunction with $\|\hat{u}_1\|_2 = 1$ and $\hat{u}_1 \geq 0$ in Ω.

Proposition 2. *Let* $V = \left\{u \in X_0 : \int_\Omega \hat{u}_1 u \; dx = 0\right\}$ *and set*

$$\hat{\lambda}_V = \inf\{\Psi(u) : u \in M \cap V\}.$$

Then $\hat{\lambda}_1 < \hat{\lambda}_V$.

Proof. First of all, it is clear from the definition above that $\hat{\lambda}_1 \leq \hat{\lambda}_V$.

Suppose by contradiction that $\hat{\lambda}_1 = \hat{\lambda}_V$. Then we can find a sequence $(u_n)_n \subset M \cap V$ such that $\Psi(u_n) \to \hat{\lambda}_V = \hat{\lambda}_1$. By (8) we have

$$\|u_n\|^2_{X_0}(1 - \epsilon\|\beta\|_q) \leq \Psi(u_n) + c(\epsilon)\|\beta\|_q\|u_n\|_2^2 \to \hat{\lambda}_1 + c(\epsilon)\|\beta\|_q,$$

hence, $(u_n)_n \subset X_0$ is bounded, and so we may assume

$$u_n \rightharpoonup u \text{ in } X_0 \text{ and } u_n \to u \text{ in } L^{2q'}(\Omega). \tag{10}$$

Exploiting the sequential weak lower semicontinuity of Ψ, by (10) and since $u \in M \cap V$, we have

$$\hat{\lambda}_1 \leq \Psi(u) \leq \liminf_{n\to\infty} \Psi(u_n) = \hat{\lambda}_1 = \hat{\lambda}_V,$$

and hence,

$$\Psi(u) = \hat{\lambda}_1.$$

By Proposition 1 this implies that $u = \pm\sigma\hat{u}_1$ for some $\sigma > 0$, a contradiction to the fact that $u \in M \cap V$. Thus $\hat{\lambda}_1 < \hat{\lambda}_V$. □

Proof of Theorem 2. The first part is contained in Proposition 1. Then, solving (P_λ) is equivalent to solving the eigenvalue problem

$$\begin{cases} \mathfrak{L}_K u + \tilde{\beta}(x)u = \Lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{$\tilde{P}_\Lambda$}$$

where $\tilde{\beta} = \beta - \hat{\lambda}_1 + 1$ and $\Lambda = \lambda - \hat{\lambda}_1 + 1$. Thus, in order to show that $(\tilde{P}_\Lambda)$ has a diverging sequence of eigenvalues, we apply the classical genus theorem in the form of [15, Theorem 9.26]. Hence, set

$$\phi(u) = \int_\Omega u^2 dx, \;\; \psi(u) = \Psi(u) - (\hat{\lambda}_1 - 1)\int_\Omega u^2 dx$$

and

$$\mathcal{M} := \{u \in X_0 \;\; : \;\; \psi(u) = 1\}.$$

By definition of $\hat{\lambda}_1$, it is readily seen that $\int u^2 \leq 1$ if $u \in \mathcal{M}$. As a consequence, by (8) we get that, if $u \in \mathcal{M}$, then

$$\|u\|^2_{X_0}(1 - \epsilon\|\beta\|_q) \leq 1 + |\hat{\lambda}_1 - 1| + c(\epsilon)\|\beta\|_q.$$

Hence, $\mathcal{M}$ is bounded. The other assumptions of Motreanu et al[15, Theorem 9.26] are easily verified, and so there exists a sequence $\{(\Lambda_n, u_n)\}_n$ of solutions to $(\tilde{P}_\Lambda)$ with $\Lambda_n \neq 0$ and $1/\Lambda_n \to 0$, $\int u_n \to 0$ as $n \to \infty$. In particular,

$$1 = \psi(u_n) = (\Lambda_n - \hat{\lambda}_1 + 1)\int_\Omega u_n^2 dx \;\; \text{for all } n \in \mathbb{N},$$

which implies that $\Lambda_n \to +\infty$ as $n \to \infty$, and so

$$\hat{\lambda}_n \to +\infty \quad \text{as} \quad n \to \infty,$$

as claimed. □

4 | MOUNTAIN PASS SOLUTIONS BELOW THE FIRST EIGENVALUE

In this section, we study the following nonlinear fractional problem

$$\begin{cases} \mathfrak{L}_K u + \beta(x)u = f(x, u(x)) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{P}$$

where, as before, $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$ and β may be sign changing. As for f, we shall assume

Hypothesis 1. *$f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(x, 0) = 0$ for a.e. $x \in \Omega$ and*

(1) *$|f(x,t)| \leq a(x)(1 + |t|^{p-1})$ for a.e $x \in \Omega$, all $t \in \mathbb{R}$ with $a \in L^\infty(\Omega)_+ = \{a \in L^\infty\{\Omega\} : a \geq 0 \text{ a.e. in } \Omega\}$, $p \in (2, 2^\star)$;*

(2) *if $F(x,t) = \int_0^t f(x,w)\, dw$, then*

$$\lim_{t\to\pm\infty} \frac{F(x,t)}{t^2} = \infty \text{ uniformly for a.e } x \in \Omega;$$

(3) *if $\xi(x,t) = f(x,t)t - 2F(x,t)$, then there exists $\beta^* \in L^1(\Omega)_+$ such that*

$$\xi(x,t) \leq \xi(x,y) + \beta^*(x) \text{ for a.e. } x \in \Omega \text{ and all } 0 \leq t \leq y, \text{ or } y \leq t \leq 0;$$

(4) *there exist $\vartheta_0 \in L^\infty(\Omega)$ and $\eta_0 > 0$ such that*

$$-\eta_0 \leq \liminf_{t\to 0} \frac{f(x,t)}{t} \leq \limsup_{t\to 0} \frac{f(x,t)}{t} \leq \vartheta_0(x)$$

uniformly for a.e $x \in \Omega$, where ϑ_0 is such that one of the following conditions holds:

(i) *$\beta^+ \in L^\infty_{loc}(\Omega)$ or $K(x) = \frac{1}{|x|^{N+2s}}$ and $\vartheta_0 \leq \hat{\lambda}_1$, $\vartheta_0 \neq \hat{\lambda}_1$;*

(ii) *$\vartheta_0 < \hat{\lambda}_1$ a.e. in Ω.*

Of course, the requirement in Hypothesis 1 (4)(i) ensures that the first eigenfunction is strictly positive in Ω, see Remark 2.

Remark 5. Hypothesis 4.1(3) was introduced in[3] to replace the stronger Ambrosetti–Rabinowitz condition.

Now, we introduce the functional $\varphi : X_0 \to \mathbb{R}$ defined as

$$\varphi(u) = \frac{1}{2}\Psi(u) - \int_\Omega F(x, u(x))\, dx,$$

whose critical points are solutions of (P).

Proposition 3. *If Hypotheses 1 (1)-(3) hold and $\beta \in L^q(\Omega)$ with $q > \frac{N}{2s}$, then φ satisfies (C).*

Proof. Let $(u_n)_n \subset X_0$ be a sequence such that

$$|\varphi(u_n)| \leq M_1 \text{ for some } M_1 > 0, \text{ all } n \geq 1 \tag{11}$$

and

$$(1+\|u_n\|_{X_0})\varphi'(u_n) \to 0 \text{ in } X_0^* \text{ as } n\to\infty. \tag{12}$$

We have

$$2\varphi(u_n) - \varphi'(u_n)(u_n) = \int_\Omega [f(x,u_n)u_n - 2F(x,u_n)]\ dx.$$

By using (11) and (12), we immediately obtain that

$$\int_\Omega \xi(x,u_n)\ dx \le M_2 \text{ for all } n\ge 1. \tag{13}$$

Claim: $(u_n)_n \subset X_0$ *is bounded*. By contradiction we suppose that, up to a subsequence,

$$\|u_n\|_{X_0} \to \infty \text{ as } n\to\infty. \tag{14}$$

Let $y_n = \frac{u_n}{\|u_n\|_{X_0}}$, $n\ge 1$. Then $\|y_n\|_{X_0} = 1$ for all $n\ge 1$ and so we may assume that

$$y_n \rightharpoonup y \text{ in } X_0 \text{ and } y_n \to y \text{ in } L^p(\Omega) \text{ as } n\to\infty. \tag{15}$$

First, suppose that $y\neq 0$ and let $\Omega_0 = \{x\in\Omega : y(x)=0\}$. Then

$$|u_n(x)| \to \infty \text{ for a.e } x\in \Omega_0^c := \{x\in\Omega \ : \ x\notin\Omega_0\}.$$

Then Hypothesis 1 (2) and Fatou's Lemma imply that

$$\lim_{n\to\infty}\int_{\Omega_0^c} \frac{F(x,u_n(x))}{\|u_n\|_{X_0}^2}\ dx = \infty.$$

But

$$\int_\Omega \frac{F(x,u_n(x))}{\|u_n\|_{X_0}^2}\ dx = \int_{\Omega_0} \frac{F(x,u_n(x))}{\|u_n\|_{X_0}^2}\ dx + \int_{\Omega_0^c} \frac{F(x,u_n(x))}{\|u_n\|_{X_0}^2}\ dx,$$

and so

$$\lim_{n\to\infty}\int_\Omega \frac{F(x,u_n(x))}{\|u_n\|_{X_0}^2}\ dx = \infty. \tag{16}$$

On the other hand, from (11) we know that

$$\int_\Omega \frac{F(x,u_n(x))}{\|u_n\|_{X_0}^2}\ dx \le M_3 \text{ for some } M_3 \text{ and all } n\ge 1,$$

which contradicts (16).

Now suppose that $y=0$. We fix $\eta>0$ and define

$$v_n = (2\eta)^{\frac{1}{2}} y_n \in X_0 \text{ for all } n\ge 1.$$

Since

$$v_n \to 0 \text{ in } L^p(\Omega),$$

we have

$$\int_\Omega F(x,v_n)\ dx \to 0 \text{ as } n\to\infty. \tag{17}$$

By (14), we can find $n_0\ge 1$ such that

$$0 < (2\eta)^{\frac{1}{2}}\frac{1}{\|u_n\|_{X_0}} \le 1 \text{ for all } n\ge n_0. \tag{18}$$

Let $\zeta_n \in [0,1]$ be such that

$$\varphi(\zeta_n u_n) = \max_{0\le\zeta\le1} \varphi(\zeta u_n).$$

From (18) it follows that

$$\varphi(\zeta_n u_n) \ge \varphi(v_n) = \eta\Psi(y_n) - \int_\Omega F(x, v_n)\ dx \text{ for all } n \ge n_0. \tag{19}$$

As we have just seen,

$$\left|\int_\Omega \beta u^2\ dx\right| \le \|\beta\|_q \|u\|_{2q'}^2.$$

Again by Theorems 3, $X_0 \hookrightarrow L^{2q'}(\Omega) \hookrightarrow L^2(\Omega)$ and the first embedding is compact. By Ehrling's inequality, given $\epsilon > 0$ we can find $c(\epsilon) > 0$ such that

$$\|u\|_{2q'}^2 \le \epsilon\|u\|_{X_0}^2 + c(\epsilon)\|u\|_2^2 \text{ for all } u \in X_0.$$

Like in (8), we get

$$(1 - \epsilon\|\beta\|_q)\|u\|_{X_0}^2 \le \Psi(u) + c(\epsilon)\|\beta\|_q\|u\|_2^2. \tag{20}$$

Now use (20) in (19), so that

$$\varphi(\zeta_n u_n) \ge \eta\left[(1 - \epsilon\|\beta\|_q) - c(\epsilon)\|\beta\|_q\|y_n\|_2^2\right] - \int_\Omega F(x, v_n)\ dx \tag{21}$$

for all $n \ge n_0$. Choose $\epsilon \in \left(0, 1/\|\beta\|_q\right)$ and note that

$$\|y_n\|_2^2 \to 0 \text{ as } n \to \infty, \tag{22}$$

see (15) and recall that $y = 0$. By (21), using (17) and (22), we get that

$$\liminf_{n\to\infty} \varphi(\zeta_n u_n) \ge \eta(1 - \epsilon\|\beta\|_q).$$

Since $\eta > 0$ is arbitrary, by letting $\eta \to \infty$ we conclude that

$$\varphi(\zeta_n u_n) \to \infty \text{ as } n \to \infty. \tag{23}$$

Notice that

$$\varphi(0) = 0 \text{ and } \varphi(u_n) \le M_1 \text{ for all } n \ge 1.$$

Therefore, (23) implies that there exists $n_1 \ge n_0$ such that $\zeta_n \in (0,1)$ for all $n \ge n_1$; hence,

$$\frac{d}{d\zeta}\varphi(\zeta u_n)|_{\zeta=\zeta_n} = 0 \text{ for all } n \ge n_1,$$

and so

$$\Psi(\zeta_n u_n) = \int_\Omega f(x, \zeta_n u_n)\zeta_n u_n\ dx \text{ for all } n \ge n_1. \tag{24}$$

Using Hypothesis 1(3) we have

$$\int_\Omega \xi(x, \zeta_n u_n)\ dx \le \int_\Omega \xi(x, u_n)\ dx + \|\beta^*\|_1 \text{ for all } n \ge n_1.$$

Using the definition of ξ, (24) and (13) we obtain

$$2\varphi(\zeta_n u_n) = \Psi(\zeta_n u_n) - 2\int_\Omega F(x, \zeta_n u_n)\ dx = \int_\Omega \xi(x, \zeta_n u_n)\ dx \le M_4 \tag{25}$$

fore some $M_4 > 0$ and all $n \geq n_1$. Comparing (23) and (25) we reach a contradiction. This proves the claim.

By the previous claim, now we may assume that

$$u_n \rightharpoonup u \text{ in } X_0 \text{ and } u_n \to u \text{ in } L^p(\Omega). \tag{26}$$

Choosing $u_n - u \in X_0$ as test function in (12), passing to the limit as $n \to \infty$ and using (26), we find

$$\lim_{n\to\infty} \langle \mathfrak{L}_K u_n, u_n - u\rangle = 0$$

which implies that $u_n \to u$ in X_0 as $n \to \infty$, and so φ satisfies (C). □

Lemma 4. *If Hypothesis 1(4) holds, then there exists $\alpha_0 > 0$ such that*

$$\Sigma(u) = \Psi(u) - \int_\Omega \vartheta_0 u^2 \, dx \geq \alpha_0 \|u\|^2_{X_0}.$$

Proof. The lines of the proof follow those of in the proof of Mugnai and Papageorgiou.[21, Lemma 18]

Of course, $\Sigma(u) \geq 0$. By contradiction, we suppose the lemma is not true. Using the 2-homogenity of Σ, we can find $(u_n)_n \subset X_0$ such that

$$\|u_n\|_{X_0} = 1 \text{ for all } n \geq 1 \text{ and } \Sigma(u_n) \to 0^+ \text{ as } n \to \infty. \tag{27}$$

We may assume

$$u_n \rightharpoonup u \text{ in } X_0 \text{ and } u_n \to u \text{ in } L^2(\Omega) \text{ as } n \to \infty. \tag{28}$$

It follows from (28) and the lower weak semicontinuity of Ψ that $\Sigma(u) \leq 0$, and so

$$\Psi(u) \leq \int_\Omega \vartheta_0 u^2 \, dx \leq \hat{\lambda}_1 \|u\|_2^2. \tag{29}$$

If $u = 0$ then from (8) applied to u_n and (28) we see that $u_n \to 0$ in X_0, a contradiction to the fact that $\|u_n\|_{X_0} = 1$ for all $n \geq 1$. Hence $u \neq 0$, but now from (29) and Proposition 1 we can deduce that $\Psi(u) = \hat{\lambda}_1 \|u\|_2^2$, and so $u = \pm\sigma\hat{u}_1$ for some $\sigma > 0$. If Hypothesis 1.4**(i)** holds, then $\hat{u}_1(x) > 0$ for a.e. $x \in \Omega$, and so from the first inequality in (29) we have $\Psi(u) < \hat{\lambda}_1 \|u\|_2^2$, again a contradiction; if 1.4(ii) holds, then the contradiction is reached using $\vartheta_0 < \hat{\lambda}_1$ and $u \neq 0$. The lemma is thus proved. □

Now, we want to prove the existence of nontrivial solutions of constant sign. For this, we introduce the following truncations-perturbations of the reaction f:

$$\hat{f}_+(x,t) = \begin{cases} 0 & \text{if } t \leq 0, \\ f(x,t) + \gamma t & \text{if } 0 < t \end{cases} \tag{30}$$

and

$$\hat{f}_-(x,t) = \begin{cases} f(x,t) + \gamma t & \text{if } t < 0 \\ 0 & \text{if } 0 \leq t, \end{cases} \tag{31}$$

where $\gamma > c(\epsilon) \|\beta\|_q$ once ϵ is chosen (see the Proof of proposition 3). We set

$$\hat{F}_\pm(x,t) = \int_0^t \hat{f}_\pm(x,w) \, dw.$$

Then, set $\hat{\beta}(x) = \beta + \gamma$ and define

$$\hat{\Psi}(u) = \|u\|^2_{X_0} + \int_\Omega \hat{\beta}(x) u^2 \, dx \text{ for all } u \in X_0.$$

Finally, we consider the functionals $\hat{\varphi}_{\pm} : X_0 \to \mathbb{R}$ defined by

$$\hat{\varphi}_{\pm}(u) = \frac{1}{2}\hat{\Psi}(u) - \int_{\Omega} \hat{F}_{\pm}(x,u)\ dx \quad \text{for all } u \in X_0.$$

Remark 6. If we repeat the proof of Proposition 3 for the functionals $\hat{\varphi}_{\pm}$, we immediately have that they both satisfy (C).

Proposition 4. *If Hypothesis 1 holds, $\beta \in L^q(\Omega)$ with $q > \frac{N}{2s}$, then $u = 0$ is a strict local minimizer for φ and $\hat{\varphi}_{\pm}$.*

Proof. We do the proof for the functional φ, for the others being similar. By Hypotheses 1(1) and 1(4), given $\epsilon > 0$ we can find c_ϵ such that

$$F(x,t) \leq \frac{1}{2}(\vartheta_0(x) + \epsilon)t^2 + c_\epsilon |t|^p \text{ for a.e. } x \in \Omega, \text{ all } t \in \mathbb{R}, \ p > 2. \tag{32}$$

Then, for every $u \in X_0$ we have

$$\varphi(u) = \frac{1}{2}\Psi(u) - \int_{\Omega} F(x,u)\ dx \geq \frac{1}{2}\Psi(u) - \frac{1}{2}\int_{\Omega} \vartheta_0 u^2\ dx - \frac{\epsilon}{2}\|u\|_2^2 - c_\epsilon \|u\|_p^p.$$

Recalling Lemma 3, we can find $C > 0$ such that $\|u\|_2^2 \leq C\|u\|_{X_0}^2$ and $\|u\|_p^p \leq C\|u\|_{X_0}^p$. Applying these inequalities to (4) together with Lemma 4, we obtain

$$\begin{aligned}\varphi(u) &\geq \frac{1}{2}\Psi(u) - \int_{\Omega} \vartheta_0 u^2\ dz - \frac{\epsilon C}{2}\|u\|_{X_0}^2 - C_\epsilon \|u\|_{X_0}^p \\ &\geq \frac{\alpha_0 - \epsilon C}{2}\|u\|_{X_0}^2 - C_\epsilon \|u\|_{X_0}^p.\end{aligned}$$

Choosing $\epsilon \in (0, \alpha_0/C)$, we have

$$\varphi(u) \geq C_1\|u\|_{X_0}^2 - C_2\|u\|_{X_0}^p. \tag{33}$$

for some $C_1, C_2 > 0$. Since $p > 2$, from (33) we get that $u = 0$ is a strict local minimizer of φ (and similarly for the functionals $\hat{\varphi}_{\pm}$). □

Proposition 5. *If Hypothesis 1 holds and $\beta \in L^q(\Omega)$ with $q > \frac{N}{2s}$, then for every $u \in X_0 \setminus \{0\}$, we have $\varphi(\zeta u) \to -\infty$ as $\zeta \to \pm\infty$.*

Proof. By Hypothesis 1(1) and 1(2), given any $\mu > 0$ we can find $c_\mu > 0$ such that

$$F(x,t) \geq \frac{\mu}{2}t^2 - c_\mu \text{ for a.e } x \in \Omega \text{ and all } t \in \mathbb{R}.$$

Hence, for $u \in X_0$, $u \neq 0$ and $\zeta > 0$, choosing $\mu > 0$ big enough, we have

$$\varphi(\zeta u) \to -\infty \text{ as } \zeta \to \infty.$$

□

Theorem 3. *If Hypothesis 1 holds, $\beta \in L^q(\Omega)$ with $q > \frac{N}{2s}$, then problem (P) admits at least two nontrivial weak solutions $\hat{u}, \ \hat{v} \in X_0$ such that*

$$\hat{v}(x) \leq 0 \leq \hat{u}(x) \text{ a.e. in } \Omega.$$

Proof. By Proposition 4, we can find $\rho \in (0,1)$ so small that

$$\hat{\varphi}_+(0) = 0 < \inf\left\{\hat{\varphi}_+(u) \ \ \|u\|_{X_0} = \rho\right\} := \hat{m}_+. \tag{34}$$

Then (34), together with Proposition 5 and Remark 6, implies that we can use the Mountain Pass Theorem. So we can find $\hat{u} \in X_0$ such that

$$\hat{\varphi}_+(0) = 0 < \hat{m}_+ \le \hat{\varphi}_+(\hat{u}) \tag{35}$$

and

$$\hat{\varphi}'_+(\hat{u}) = 0. \tag{36}$$

From (35), we see that $\hat{u} \neq 0$, while from (36), we have

$$\mathfrak{L}_K \hat{u} + \hat{\beta}(x)\hat{u} = f_+(x, \hat{u}). \tag{37}$$

By (9)

$$\langle \hat{u}, \hat{u}^- \rangle_{X_0} = \left\langle \hat{u}^+ - \hat{u}^-, \hat{u}^- \right\rangle_{X_0} = \left\langle \hat{u}^+, \hat{u}^- \right\rangle_{X_0} - \|\hat{u}^-\|^2_{X_0} \le -\|\hat{u}^-\|^2_{X_0}. \tag{38}$$

On (37), we act with $-\hat{u}^- \in X_0$. Then together with (38), we get

$$\Psi(\hat{u}^-) + \gamma\|\hat{u}^-\|_2^2 \le \langle \hat{u}, -\hat{u}^- \rangle_{X_0} + \int_\Omega \beta(\hat{u}^-)^2 \, dx + \gamma\|\hat{u}^-\|_2^2 = 0. \tag{39}$$

From (8) with $\epsilon > 0$ small enough, we have

$$(1 - \epsilon\|\beta\|_q)\|\hat{u}^-\|^2_{X_0} - c(\epsilon)\|\beta\|_q\|\hat{u}^-\|_2^2 \le \Psi(\hat{u}^-). \tag{40}$$

Since $\gamma > c(\epsilon)\,\|\beta\|_q$, from (39), (40) and recalling that $\|\hat{u}^-\|_2^2 \le C\|\hat{u}^-\|^2_{X_0}$ (see Lemma 3) it follows that

$$C\|\hat{u}^-\|^2_{X_0} \le 0 \tag{41}$$

for some $C > 0$, which implies that

$$\hat{u} \ge 0, \;\; \hat{u} \neq 0.$$

So, (37) becomes

$$\mathfrak{L}_K \hat{u} + \beta(x)\hat{u} = f(x, \hat{u}).$$

i.e. $\hat{u}$ is a weak solution for (P).

In a similar fashion, working this time with $\hat{\varphi}_-$, we obtain another nontrivial solution $\hat{v} \in X_0$ having negative sign. □

5 | A PARAMETRIC PROBLEM WITH COMPETING NONLINEARITIES

In this section we study the following parametric nonlinear problem:

$$\begin{cases} \mathfrak{L}_K u + \beta(x)u = \lambda g(x, u(x)) + f(x, u(x)) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{E_λ}$$

$\lambda > 0$ being a parameter. Strengthening the previous assumption, here we will assume that $\beta \in L^\infty(\Omega)$. Moreover, f is a general superlinear function at ∞, while g is sublinear. In this case problem (E_λ) is an extension to the fractional setting of the problem studied in[2] and of that in[1] with more general nonlinearities and a sign changing weight in the operator. Moreover, we improve the assumptions in[2] to find the desired solutions.

Going into details, we impose the following conditions on g and f:

Hypothesis 2. $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $g(x, 0) = 0$ for a.e. $x \in \Omega$ and

(1) there exist $b \in L^\infty(\Omega)$ and $\mathcal{P} \in (2, 2^\star)$ such that

$$|g(x, t)| \le b(x)(1 + |t|^{\mathcal{P}-1}) \text{ for a.e } x \in \Omega, \;\text{ all } t \in \mathbb{R};$$

(2)

$$\lim_{t\to\pm\infty}\frac{g(x,t)}{t}=0 \text{ uniformly for a.e. } x\in\Omega;$$

(3) if $G(x,t)=\int_0^t g(x,z)\ dz$, then there exist $p,\ q\in(1,2)$, $\delta_0>0$ and $\hat{\eta}_0,\ \ \eta_0>0$ such that

$$\begin{aligned}
&0<g(x,t)t\le pG(x,t) \text{ for a.e. } x\in\Omega,\ \ 0<|t|\le\delta_0,\\
&\operatorname*{ess\,inf}_{\Omega} G(\cdot,\pm\delta_0)>0,\\
&\limsup_{t\to 0}\frac{g(x,t)}{|t|^{p-2}t}\le\hat{\eta}_0 \text{ uniformly for a.e } x\in\Omega \text{ and}\\
&\eta_0|t|^q\le g(x,t)t \text{ for a.e. } x\in\Omega,\ \ \text{all } t\in\mathbb{R}.
\end{aligned}$$

Hypothesis 3. $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that $f(x,0)=0$ for a.e. $x\in\Omega$ and

(1) there exist $a\in L^\infty(\Omega)_+$ and $r\in(2,2^\star)$ such that

$$|f(x,t)|\le a(x)(1+|t|^{r-1}) \text{ for a.e. } x\in\Omega,\ \ \text{all } t\in\mathbb{R};$$

(2)

$$\lim_{t\to\pm\infty}\frac{f(x,t)}{t}=+\infty \text{ uniformly for a.e. } x\in\Omega;$$

(3) there exists $\vartheta\in[0,\hat{\lambda}_1)$ such that

$$\lim_{t\to 0}\frac{f(x,t)}{t}=\vartheta \text{ uniformly for a.e. } x\in\Omega.$$

Now, for $\lambda>0$ set

$$\xi_\lambda(x,t)=\lambda g(x,t)t+f(x,t)t-2\lambda G(x,t)-2F(x,t),$$

where $F(x,t)=\int_0^t f(x,w)\ dw$.

Following Mugnai and Papageorgiou,[3] we introduce the following condition.

Hypothesis 4 For every $\lambda>0$, there exists $\beta_\lambda^*\in L^1(\Omega)$ such that

$$\xi_\lambda(x,t)\le\xi_\lambda(x,y)+\beta_\lambda^*(x)$$

for a.e $x\in\Omega$ and all $0\le t\le y$ or $y\le t\le 0$.

Remark 7. The condition $\operatorname{ess\,inf}_\Omega G(\cdot,\pm\delta_0)>0$ in Hypothesis 2(3) is automatically satisfied if stronger assumptions on g are required, see.[22]

Example 1. Examples of functions satisfying the growth assumptions introduced above are given by

$$g(x,t)=|t|^{p-2}t \ \text{ and }\ f_1(x,t)=|t|^{r-2}t$$

where $p=q$, or

$$f_2(x)=\begin{cases}\vartheta x-\frac{\vartheta}{2}|x|^{r-2}x & \text{if } |x|\le 1,\\ \vartheta x\left(\log|x|+\frac{1}{2}\right) & \text{if } |x|>1,\end{cases}$$

where $2<r<2^*$ and $\vartheta<\hat{\lambda}_1$. Notice that f_1 satisfies the AR–condition, while f_2 does not.

Now, we denote by $\varphi_\lambda:X_0\to\mathbb{R}$ the energy functional associated to problem (E_λ), namely,

$$\varphi_\lambda(u)=\frac{1}{2}\Psi(u)-\lambda\int_\Omega G(x,u(x))\ dx-\int_\Omega F(x,u(x))\ dx$$

for all $u \in X_0$. It is standard to see that $\varphi_\lambda \in C^1(X_0)$.

As above, by suitable truncation–perturbations of the map $t \mapsto \lambda g(x,t) + f(x,t)$, we will produce signed nontrivial solutions to (E_λ). To do that, from now on we assume that

$$\beta \in L^\infty(\Omega).$$

So, by (6), fixed $\epsilon > 0$, if $\tau > \frac{2^\star}{2^\star - 2}$, we choose $\gamma > \|\beta\|_\infty$ and

$$\gamma > c(\epsilon)\|\beta\|_\infty |\Omega|^{\frac{1}{\tau}} \geq c(\epsilon)\|\beta\|_\tau \text{ with } c(\epsilon) \geq 1.$$

Now define

$$h_\lambda^+(x,t) = \begin{cases} 0 & \text{if } t \leq 0, \\ \lambda g(x,t) + f(x,t) + \gamma t & \text{if } 0 < t \end{cases} \tag{42}$$

$$h_\lambda^-(x,t) = \begin{cases} \lambda g(x,t) + f(x,t) + \gamma t & \text{if } t < 0 \\ 0 & \text{if } 0 \leq t. \end{cases}$$

Both $h_\lambda^\pm$ are Carathéodory functions. We set

$$H_\lambda^\pm(x,t) = \int_0^t h_\lambda^\pm(x,w)\, dw$$

and consider the C^1-functionals $\varphi_\lambda^\pm : X_0 \to \mathbb{R}$ defined by

$$\varphi_\lambda^\pm(u) = \frac{1}{2}\Psi(u) + \frac{\gamma}{2}\|u\|_2^2 - \int_\Omega H_\lambda^\pm(x,u(x))\, dx \text{ for all } u \in X_0.$$

Notice that Hypotheses 2, 3, 4 imply that the map $(x,t) \mapsto \lambda g(x,t) + f(x,t)$ satisfies Hypothesis 1. Thus, by Proposition 3, we immediately have that

Proposition 6. *If Hypotheses 2, 3 and 4 hold, $\lambda > 0$ and $\beta \in L^\infty(\Omega)$, then φ_λ and $\varphi_\lambda^\pm$ satisfy (C).*

In order to prove the existence of two signed solutions, we shall apply the Mountain Pass Theorem. We start with the next two propositions, where we show the validity of the mountain pass geometry for $\varphi_\lambda^\pm$, provided that $\lambda > 0$ is small enough.

Proposition 7. *Assume Hypotheses 2, 3 and 4 hold, $\lambda > 0$ and $\beta \in L^\infty(\Omega)$. Then:*

1. *There exists $\lambda_+^* > 0$ such that for all $\lambda \in (0, \lambda_+^*)$ there exists $\rho_\lambda^+ > 0$ such that*

$$\inf\left\{\varphi_\lambda^+(u) : \|u\|_{X_0} = \rho_\lambda^+\right\} := m_\lambda^+ > 0.$$

2. *There exists $\lambda_-^* > 0$ such that for all $\lambda \in (0, \lambda_-^*)$ there exists $\rho_\lambda^- > 0$ for which we have*

$$\inf\left\{\varphi_\lambda^-(u) : \|u\|_{X_0} = \rho_\lambda^-\right\} := m_\lambda^- > 0.$$

Proof. Without loss of generality, we assume $\mathcal{P} \leq r$ (otherwise r is replaced by $\mathcal{P}$ in the calculations below).

Hypotheses 2 and 3 imply that given $\tilde\vartheta > \vartheta > 0$ with $\gamma > c(\epsilon)\|\beta\|_\tau + \tilde\vartheta$, we can find $C = C(\tilde\vartheta) > 0$ such that

$$H_\lambda^+(x,t) \leq \frac{\tilde\vartheta}{2}(t^+)^2 + \lambda C(t^+)^p + C(1+\lambda)(t^+)^r \text{ for a.e. } x \in \Omega, \text{ all } t \in \mathbb{R}. \tag{43}$$

Then, for all $u \in X_0$, using (8) and Theorem 3 we have

$$\begin{aligned}\varphi_\lambda^+(u) &\geq \frac{(1-\epsilon\|\beta\|_\tau)}{2}\|u\|_{X_0}^2 - \frac{c(\epsilon)\|\beta\|_\tau}{2}\|u\|_2^2 + \frac{\gamma}{2}\|u\|_2^2 - \frac{\tilde{\vartheta}}{2}\|u\|_2^2 \\ &\quad - \lambda C\|u\|_p^p - (1+\lambda)C\|u\|_r^r \\ &\geq \frac{1}{2}(1-\epsilon\|\beta\|_\tau)\|u\|_{X_0}^2 + \frac{1}{2}(-c(\epsilon)\|\beta\|_\tau + \gamma - \tilde{\vartheta})\|u\|_2^2 \\ &\quad - \lambda B\|u\|_{X_0}^p - (1+\lambda)D\|u\|_{X_0}^r \\ &\geq \left(A - \lambda B\|u\|_{X_0}^{p-2} - D(1+\lambda)\|u\|_{X_0}^{r-2}\right)\|u\|_{X_0}^2,\end{aligned} \tag{44}$$

for some $A, B, D > 0$.

Now, consider the function

$$k_\lambda(y) = \lambda B y^{p-2} + D(1+\lambda)y^{r-2} \text{ for all } y \in \mathbb{R}.$$

Of course, $k_\lambda \in C^1(0,\infty)$ and since $p < 2 < r$ (see Hypotheses 2 and 3), we have

$$k_\lambda(y) \to \infty \text{ as } y \to 0^+ \text{ and as } y \to \infty.$$

Thus, there exists $y_0 \in (0,\infty)$ such that

$$k_\lambda(y_0) = \min\{k_\lambda(y) : y > 0\}.$$

Then $k_\lambda'(y_0) = 0$, so that $\lambda B(2-p) = D(1+\lambda)(r-2)y_0^{r-p}$, and so

$$y_0(\lambda) = \left[\frac{\lambda B(2-p)}{D(1+\lambda)(r-2)}\right]^{\frac{1}{r-p}}.$$

Since

$$1 + \frac{p-2}{r-p} = \frac{r-2}{r-p} > 0,$$

we get that $k_\lambda(y_0) \to 0^+$ as $\lambda \to 0^+$ and so there exists $\lambda_+^* > 0$ such that for every $\lambda \in (0, \lambda_+^*)$ we have

$$k_\lambda(y_0) < A.$$

So, from (44) it follows that

$$\inf\left\{\varphi_\lambda^+(u) : \|u\|_{X_0} = \rho_\lambda^+ = y_0(\lambda)\right\} = m_\lambda^+ \geq A - k_\lambda(y_0) > 0$$

for all $\lambda \in (0, \lambda_+^*)$.

In a completely analogous way, we show the corresponding result for φ_λ^-, valid for all $\lambda \in (0, \lambda_-^*)$. □

For the next result, we set

$$\lambda^* = \min\left\{\lambda_+^*, \lambda_-^*\right\}.$$

Proposition 8. *If Hypotheses 2, 3, 4 hold, $\lambda \in (0, \lambda^*)$ and $\beta \in L^\infty(\Omega)$, then for every $u \in X_0$, with $u \geq 0$ and $\|u\|_2 = 1$, we have $\varphi_\lambda^+(\zeta u) \to -\infty$ as $\zeta \to \infty$.*

Proof. Hypotheses 2(1) and 2(2) imply that, given $\sigma > 0$, there exists $C_1 = C_1(\sigma) > 0$ such that

$$\lambda G(x,t) \geq -\frac{\sigma}{2}t^2 - C_1 \text{ for a.e. } x \in \Omega, \text{ all } t \in \mathbb{R}, \text{ all, } \lambda \in (0,\lambda^*). \tag{45}$$

Similarly, Hypotheses 3(1) and 3(2) imply that, given $\mu > 0$, we can find $C_2 = C_2(\mu) > 0$ such that

$$F(x,t) \geq \frac{\mu}{2}t^2 - C_2 \text{ for a.e. } x \in \Omega, \text{ all } t \in \mathbb{R}. \tag{46}$$

Let $u \in X_0$, with $u \geq 0$ and $\|u\|_2 = 1$, and let $\zeta > 0$. Then, from (42), (45) and (46), we have

$$\varphi_\lambda^+(\zeta u) \leq \frac{\zeta^2}{2}\left[\|u\|_{X_0}^2 + (\|\beta\|_\infty + \sigma - \mu + C)\right] \text{ (since } \|u\|_2 = 1) \tag{47}$$

for some $C > 0$.

Since $\sigma > 0$ and $\mu > 0$ are arbitrary, we can choose $\sigma > 0$ so small and $\mu > 0$ so large that $\mu - \sigma > \|u\|_{X_0}^2 + \|\beta\|_\infty + C$. Then, from (47), we infer that

$$\varphi_\lambda^+(\zeta u) \to -\infty \text{ as } \zeta \to \infty.$$

□

Remark 8. In an analogous way, we have that if $u \in X_0$, with $u \leq 0$ and $\|u\|_2 = 1$, then

$$\varphi_\lambda^-(\zeta u) \to -\infty \text{ as } \zeta \to \infty.$$

Now we can show the existence of two nontrivial constant sign solutions.

Proposition 9. *If Hypotheses 2, 3 and 4 hold, $\lambda \in (0, \lambda^*)$ and $\beta \in L^\infty(\Omega)$, then problem (E_λ) admits at least two nontrivial weak solution such that*

$$v_0(x) \leq 0 \leq u_0(x) \text{ for a.e } x \in \Omega.$$

Proof. We do the proof for the functional φ_λ^+. By Proposition 7, for every $\lambda \in (0, \lambda^*)$ it is possible to find $\rho_\lambda^+ > 0$ such that

$$m_\lambda^+ = \inf\left\{\varphi_\lambda^+(u) : \|u\|_{X_0} = \rho_\lambda^+\right\} > 0. \tag{48}$$

Thanks to (48) and Propositions 6 and 8, we can apply the Mountain Pass Theorem. So we can find $u_0 \in X_0$ such that

$$\varphi_\lambda^+(0) = 0 < m_\lambda^+ \leq \varphi_\lambda^+(u_0) \tag{49}$$

and

$$\varphi_\lambda^{+\prime}(u_0) = 0. \tag{50}$$

The inequalities in (49) tell us that $u_0 \neq 0$, while from (50) we have

$$\mathfrak{L}_K u_0 + (\beta(x) + \gamma)u_0 = h_\lambda^+(x, u_0). \tag{51}$$

Like we have already done in Proposition 3, thanks to (9) we have

$$\left\langle u_0, u_0^-\right\rangle_{X_0} = \left\langle u_0^+ - u_0^-, u_0^-\right\rangle_{X_0} = \left\langle u_0^+, u_0^-\right\rangle_{X_0} - \|u_0^-\|_{X_0}^2 \leq -\|u_0^-\|_{X_0}^2. \tag{52}$$

Acting on (51) with $-u_0^-$ and using (52), we obtain

$$\Psi(u_0^-) + \gamma\|u_0^-\|_2^2 \leq \left\langle u_0, -u_0^-\right\rangle_{X_0} + \int_\Omega \beta(u_0^-)^2\, dx + \gamma\|u_0^-\|_2^2 = 0. \tag{53}$$

Recalling now (8) with $\epsilon > 0$ small enough, we have

$$(1 - \epsilon\|\beta\|_\tau)\|u_0^-\|_{X_0}^2 - c(\epsilon)\|\beta\|_\tau\|u_0^-\|_2^2 \leq \Psi(u_0{}^-), \tag{54}$$

with $\gamma > c(\epsilon)\,\|\beta\|_\tau$. Hence, by (53), (54) and recalling that $\|u_0{}^-\|_2^2 \leq C\|u_0{}^-\|_{X_0}^2$ (see 3) it follows that

$$\tilde{C}\|u_0^-\|_{X_0}^2 \leq 0$$

for some $\tilde{C} > 0$, which implies that

$$u_0 \geq 0,\ \ u_0 \neq 0.$$

So, (51) becomes

$$\mathfrak{L}_K u_0 + \beta(x)u_0 = f(x, u_0) + \lambda g(x, u),$$

that is, u_0 is weak solution for (E_λ).

Analogously, working with $\hat{\varphi}_\lambda^-$, we obtain the other nontrivial solution $v_0 \in X_0$ such that $v_0 \le 0$. □

In the next proposition we produce a third nontrivial solution for (E_λ) when $\lambda \in (0, \lambda^*)$.

Proposition 10. *If Hypotheses 2, 3, 4 hold, $\lambda \in (0, \lambda^*)$ and $\beta \in L^\infty(\Omega)$, then problem (E_λ) has a third nontrivial weak solution $y_0 \in [u_0, v_0]$.*

Proof. Let u_0 and v_0 the two constant sign solutions found in Proposition 9. Choosing γ as before, we consider the following truncation perturbation of the reaction in problem (E_λ):

$$d_\lambda(x,t) = \begin{cases} \lambda g(x, v_0(x)) + f(x, v_0(x)) + \gamma v_0(x), & \text{if } t < v_0(x), \\ \lambda g(x,t)) + f(x,t) + \gamma t, & \text{if } v_0(x) < t < u_0(x), \\ \lambda g(x, u_0(x)) + f(x, u_0(x)) + \gamma u_0(x), & \text{if } t > u_0(x). \end{cases} \tag{55}$$

Of course, d_λ is a Carathéodory function. Now, set $D_\lambda = \int_0^x d_\lambda(x,w)\ dw$ and consider the C^1-functional $\Xi_\lambda : X_0 \to \mathbb{R}$ defined by

$$\Xi_\lambda(u) = \frac{1}{2}\Psi(u) + \frac{\gamma}{2}\|u\|_2^2 - \int_\Omega D_\lambda(x, u(x))\ dx \text{ for all } u \in X_0.$$

By (8), since $\gamma > \|\beta\|_\infty$, we get that

$$\Psi(u) + \gamma\|u\|_2^2 \ge C\|u\|_{X_0}^2 \tag{56}$$

for some $C > 0$.

From (56) and (55), it is clear that Ξ_λ is coercive. Moreover, it is sequentially weakly lower semicontinuous. So, by the Weierstrass Theorem, we can find $y_0 \in X_0$ such that

$$\Xi_\lambda(y_0) = \inf\{\Xi_\lambda(u) : u \in X_0\}. \tag{57}$$

By Hypothesis 3 (3), given $\epsilon > 0$ we can find $\delta = \delta(\epsilon) > 0$ such that

$$F(x,t) \ge \frac{\vartheta - \epsilon}{2} t^2 \text{ for a.e. } x \in \Omega, \text{ all } |t| \le \delta. \tag{58}$$

Recalling Remark 3, for $\zeta \in (0,1)$ small enough, we have that $\zeta\hat{u}_1 \in (0, \delta]$ for all $x \in \Omega$. Then

$$\Xi_\lambda(\zeta\hat{u}_1) \le \frac{\zeta^2}{2}\left[\hat{\lambda}_1 + \gamma - \vartheta + \epsilon\right] - \frac{\lambda\eta_0\zeta^q}{q}\|\hat{u}_1\|_q^q,$$

see (4), (42), (58) and Hypothesis 2 (3).

Since by Hypothesis 2 (3) $q < 2$, by choosing $\zeta \in (0,1)$ even smaller if necessary, we have

$$\Xi_\lambda(\zeta\hat{u}_1) < 0,$$

so that from (57)

$$\Xi_\lambda(y_0) < 0 = \Xi_\lambda(0), \tag{59}$$

and hence $y_0 \ne 0$.

From 57 we have $\Xi'_\lambda(y_0) = 0$, that is

$$\mathfrak{L}_K y_0 + (\beta(x) + \gamma)\, y_0 = d_\lambda(x, y_0). \tag{60}$$

On (60) we act with $(v_0 - y_0)^+ \in X_0$. Then, by (55) we get

$$\begin{aligned}\left\langle y_0, (v_0 - y_0)^+ \right\rangle_{X_0} &+ \int_\Omega (\beta(x) + \gamma)\, y_0 (v_0 - y_0)^+ \; dx \\ &= \int_\Omega d_\lambda(x, y_0)(v_0 - y_0)^+ \; dx = \int_\Omega d_\lambda(x, v_0)(v_0 - y_0)^+ \; dx \\ &= \left\langle v_0, (v_0 - y_0)^+ \right\rangle_{X_0} + \int_\Omega (\beta(x) + \gamma)\, v_0 (v_0 - y_0)^+ \; dx,\end{aligned}$$

hence

$$\left\langle v_0 - y_0, (v_0 - y_0)^+ \right\rangle_{X_0} + \int_\Omega (\beta(x) + \gamma)\, (v_0 - y_0)(v_0 - y_0)^+ \; dx = 0.$$

Then, recalling the choice of γ, there exists $\tilde{C} > 0$ such that

$$\left\langle v_0 - y_0, (v_0 - y_0)^+ \right\rangle_{X_0} + \tilde{C} \int_\Omega (v_0 - y_0)\,(v_0 - y_0)^+ \; dx \leq 0,$$

and so, using (9) in the first inequality, we get

$$\begin{aligned}\|(v_0 - y_0)^+\|_{X_0}^2 &\leq \|(v_0 - y_0)^+\|_{X_0}^2 - \left\langle (v_0 - y_0)^-, (v_0 - y_0)^+ \right\rangle_{X_0} \\ &\quad + \tilde{C} \int_\Omega \left[(v_0 - y_0)^+\right]^2 \\ &= \left\langle v_0 - y_0, (v_0 - y_0)^+ \right\rangle_{X_0} + \tilde{C} \int_\Omega (v_0 - y_0)(v_0 - y_0)^+ \leq 0.\end{aligned}$$

Thus, we get that $v_0 \leq y_0$ in Ω.

In an analogous way, acting on (60) with $(y_0 - u_0)^+ \in X_0$ and repeating similar calculations, we obtain $y_0 \leq u_0$. Putting together the two inequalities, we have $y_0 \in [v_0, u_0] = \{u \in X_0 \,:\, v_0(x) \leq u(x) \leq u_0(x) \;\; \forall x \in \Omega\}$.

Then (60) reads

$$\mathfrak{L}_K y_0 + \beta(x) y_0 = \lambda g(x, y_0) + f(x, y_0),$$

that is, y_0 is a weak solution of problem (E_λ). □

In conclusion, summarizing the results above, we can state the following multiplicity theorem:

Theorem 4. *If Hypotheses 2, 3, 4 hold and $\beta \in L^\infty(\Omega)$, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem (E_λ) has at least three nontrivial weak solutions $u_0,\; v_0,\; y_0 \in C^s(\bar{\Omega}) \setminus \{0\}$ such that*

$$\begin{aligned}&u_0 \geq 0 \;\; \textit{for a.e.} \;\; x \in \Omega, \\ &v_0 \leq 0 \;\; \textit{for a.e.} \;\; x \in \Omega, \\ &y_0 \in [u_0, v_0]\,.\end{aligned}$$

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CONFLICT OF INTEREST

There are no conflicts of interest to this work.

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