Corrigendum and improvements to "*Carleman estimates, observability inequalities and null controllability for interior degenerate non smooth parabolic equations*, and its consequences

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Abstract

This paper is a corrigendum of a hypothesis introduced in [**2**] and used again in [**1**] and [**3**]. We give here the corrected proofs of the concerned results, improving most of them.

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1. Introduction

In Chapter 5 of [**2**] there is a mistake in one assumption, which do not affect the results of the paper in the previous chapters. Precisely, the assumption in equation (5.2) has to be replaced by equation (5.6) therein. This replacement propagates in Hypotheses 5.2 and 5.3, as well as in Sections 4 and 5 of [**1**] and in Hypothesis 4.1(ii) of [**3**]. For the readers' convenience, we write here all the proofs of the involved results, though they are very similar. We remark that in all cases, replacing the correct assumptions makes all the written results true, and in most cases the proofs we give here are simpler than before. Each section follows the notations and the enumeration of the corresponding paper.

The authors apologize for any inconvenience caused.

2. Correction to [2]

Hypothesis 5.1 must be replaced by

HYPOTHESIS 5.1. The subset ω is such that

- it is an interval which contains the degeneracy point:

$$\omega = (\alpha, \beta) \subset (0,1) \text{ is such that } x_0 \in \omega, \tag{0.1}$$

or

- $\omega = \omega_1 \cup \omega_2$, where ω_i, $i = 1, 2$, are intervals each of them lying on different sides of the degeneracy point, more precisely:

$$\omega_i = (\lambda_i, \beta_i) \subset (0,1),\ i = 1, 2, \text{ and } \beta_1 < x_0 < \lambda_2. \tag{0.2}$$

REMARK 1. Observe that, if (0.1) holds, we can find two subintervals $\omega_1 = (\lambda_1, \beta_1) \subset\subset (\alpha, x_0), \omega_2 = (\lambda_2, \beta_2) \subset\subset (x_0, \beta)$.

As a consequence, Hypotheses 5.2 and 5.3 now read as follows:

HYPOTHESIS 5.2. Hypothesis 4.1 is satisfied. Moreover, if Hypothesis 1.1 holds, we assume that there exist two functions $\mathfrak{g} \in L^\infty_{\text{loc}}([-\beta_1, 1] \setminus \{x_0\})$, $\mathfrak{h} \in W^{1,\infty}_{\text{loc}}([-\beta_1, 1] \setminus \{x_0\}, L^\infty(0,1))$ and two constants $\mathfrak{g}_0$, $\mathfrak{h}_0 > 0$ such that $\mathfrak{g}(x) \ge \mathfrak{g}_0$ for a.e. x in $[-\beta_1, 1]$ and

$$-\frac{\tilde{a}'(x)}{2\sqrt{\tilde{a}(x)}}\left(\int_x^B \mathfrak{g}(t)dt + \mathfrak{h}_0\right) + \sqrt{\tilde{a}(x)}\mathfrak{g}(x) = \mathfrak{h}(x, B) \quad \text{for a.e. } x \in [-\beta_1, 1], B \in [0,1] \tag{0.3}$$

with $x < B < x_0$ or $x_0 < x < B$, where

$$\tilde{a}(x) := \begin{cases} a(x), & x \in [0,1], \\ a(-x), & x \in [-1, 0]. \end{cases}$$

Here β_1 is the number appearing in (0.2) or in Remark 1.

HYPOTHESIS 5.3. Hypothesis 4.2 is satisfied. Moreover, if Hypothesis 1.1 holds, then there exist two functions $\mathfrak{g} \in L^\infty_{\text{loc}}([-\beta_1, 1] \setminus \{x_0\})$, $\mathfrak{h} \in W^{1,\infty}_{\text{loc}}([-\beta_1, 1] \setminus \{x_0\}, L^\infty(0,1))$ and two constants $\mathfrak{g}_0$, $\mathfrak{h}_0 > 0$ such that $\mathfrak{g}(x) \ge \mathfrak{g}_0$ for a.e. x in $[-\beta_1, 1]$ and

$$\frac{\tilde{a}'(x)}{2\sqrt{\tilde{a}(x)}}\left(\int_x^B \mathfrak{g}(t)dt + \mathfrak{h}_0\right) + \sqrt{\tilde{a}(x)}\mathfrak{g}(x) = \mathfrak{h}(x, B) \quad \text{for a.e. } x \in [-\beta_1, 1], B \in [0,1] \tag{0.4}$$

with $x < B < x_0$ or $x_0 < x < B$.

Using new Hypothesis 5.1, Corollaries 5.1 and 5.2 become part of Proposition 5.1 and Proposition 5.3, and Theorems 5.2 and 5.4 part of Theorems 5.1 and 5.3, respectively. Hence in subsections 5.1 and 5.2 we find only new Proposition 5.1, Theorem 5.1, Proposition 5.3 and Theorem 5.3 with the new hypotheses. As a consequence, in the statement of Proposition 6.2 condition (5.2) has to be replaced by (2) above. The new proofs, which are adaptations of those in [**2**], are presented below.

The divergence case

Proposition 5.1 (and consequent Theorem 5.1) has the same statement with new Hypotheses 5.1 and 5.2. Lemma 5.1, which is independent of (0.2), obviously holds, but the proof can be improved in the last part, starting from (5.20).

PROOF OF THE SECOND PART OF LEMMA 5.1. Take ω_1, ω_2 as in Remark 1. If v solves (5.7), define, also for later uses,

$$W(t,x) := \begin{cases} -v(t,-x), & x \in [-1,0], \\ v(t,x), & x \in [0,1], \\ -v(t,2-x), & x \in [1,2]. \end{cases} \tag{0.5}$$

Take $\rho : [-1,1] \to [0,1]$ as defined after (5.20) and set $Z = \rho W$. Then Z solves

(0.6)
$$\begin{cases} Z_t + (\tilde{a} Z_x)_x = \bar{h}, & (t,x) \in (0,T) \times (-\beta_1, \beta_1), \\ Z(t,-\beta_1) = Z(t,\beta_1) = Z_x(t,-\beta_1) = Z_x(t,\beta_1) = 0, & t \in (0,T), \end{cases}$$

with $\bar{h} = (\tilde{a}\rho_x W)_x + \tilde{a}\rho_x W_x$, supported in $\Big[-\frac{\lambda_1+\beta_1}{2}, -\lambda_1\Big] \cup \Big[\lambda_1 \frac{\lambda_1+\beta_1}{2}\Big]$. Since (0.3) implies that $\tilde{a}'$ is bounded far from x_0, by [**2**, Theorem 3.1] and [**2**, Remark 7] we get

(0.7)
$$\begin{aligned} &\int_0^T \int_{-\beta_1}^{\beta_1} \left[s\Theta (Z_x)^2 e^{2s\Phi} + s^3\Theta^3 Z^2 e^{2s\Phi} \right] dxdt \le C \int_0^T \int_{-\beta_1}^{\beta_1} e^{2s\Phi} \bar{h}^2 dxdt \\ &\le C \int_0^T \int_{-\frac{\lambda_1+\beta_1}{2}}^{-\lambda_1} e^{2s\Phi}(W^2 + (W_x)^2) dxdt + C \int_0^T \int_{\lambda_1}^{\frac{\lambda_1+\beta_1}{2}} e^{2s\Phi}(W^2 + (W_x)^2) dxdt \\ &\le C \int_0^T \int_{-\beta}^{-\alpha} W^2 dxdt + C \int_0^T \int_\omega W^2 dxdt \le C \int_0^T \int_\omega v^2 dxdt, \end{aligned}$$

for some positive constants C, where Φ is the weight defined in (3.3) of [**2**] for the interval $[-\beta_1, \beta_1]$. Now, one can prove as in [**2**] that there exists a positive constant k, for example

$$k = \max \left\{ \max_{[0,\lambda_1]} a, \frac{(x_0)^2}{a(0)} \right\},$$

such that

$$a(x) e^{2s\varphi(t,x)} \le k e^{2s\Phi(t,x)} \quad \text{and} \quad \frac{(x-x_0)^2}{a(x)} e^{2s\varphi(t,x)} \le k e^{2s\Phi(t,x)}$$

for every $(t,x) \in [0,T] \times [0,\lambda_1]$. Thus, by (0.7), one has

$$\begin{aligned} &\int_0^T \int_0^{\lambda_1} \Big(s\Theta a (v_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} v^2 \Big) e^{2s\varphi} dxdt \\ &= \int_0^T \int_0^{\lambda_1} \Big(s\Theta a (Z_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} Z^2 \Big) e^{2s\varphi} dxdt \\ &\le k \int_0^T \int_{-\beta_1}^{\beta_1} (s\Theta (Z_x)^2 e^{2s\Phi} dxdt + s^3\Theta^3 Z^2 e^{2s\Phi}) dxdt \le kC \int_0^T \int_\omega v^2 dxdt. \end{aligned}$$

Hence, Lemma 5.1 follows. □

As said, Lemma 5.2 holds assuming (0.2) in place of (5.2).

Proof of Lemma 5.2. Fix $\bar\lambda_i, \bar\beta_i \in (\lambda_i, \beta_i)$, $i = 1, 2$, with $\bar\lambda_i < \bar\beta_i$ and consider a smooth function $\xi : [0, 1] \to [0, 1]$ such that

$$\xi(x) = \begin{cases} 0 & x \in [0, \bar\lambda_1], \\ 1 & x \in [\tilde\lambda_1, \tilde\lambda_2], \\ 0 & x \in [\bar\beta_2, 1], \end{cases} \tag{0.8}$$

where $\tilde\lambda_i = (\bar\lambda_i + \bar\beta_i)/2$, $i = 1, 2$. Then, define $w := \xi v$, where v is any fixed solution of (5.7), so that w satisfies

$$\begin{cases} w_t + (aw_x)_x = f, & (t, x) \in Q_T, \\ w(t, 0) = w(t, 1) = 0, & t \in (0, T), \end{cases}$$

with $f^2 = ((a\xi_x v)_x + a\xi_x v_x)^2 \le C(v^2 + (v_x)^2)\chi_{\hat\omega}$, where $\hat\omega = (\bar\lambda_1, \tilde\lambda_1) \cup (\tilde\lambda_2, \bar\beta_2)$. By applying [**2**, Theorem 4.1], we have

$$\int_0^T\int_0^1 \Big(s\Theta a(w_x)^2 + s^3\Theta^3 \frac{(x - x_0)^2}{a}\; w^2\Big) e^{2s\varphi}\, dxdt \le C\int_0^T\int_0^1 e^{2s\varphi} f^2 dxdt, \tag{0.9}$$

for all $s \ge s_0$. Hence, by [**2**, Proposition 5.2] we find

$$\begin{aligned} &\int_0^T\int_{\tilde\lambda_1}^{\tilde\lambda_2} \left(s\Theta a(v_x)^2 + s^3\Theta^3 \frac{(x - x_0)^2}{a} v^2\right) e^{2s\varphi} dxdt \\ &\le \int_0^T\int_0^1 \Big(s\Theta a(w_x)^2 + s^3\Theta^3 \frac{(x - x_0)^2}{a} w^2\Big) e^{2s\varphi}\, dxdt \\ &\le C\int_0^T\int_{\hat\omega} e^{2s\varphi}(v^2 + (v_x)^2) dxdt \le C\int_0^T\int_\omega v^2 dxdt. \end{aligned} \tag{0.10}$$

Analogously, also for future purposes, we define a smooth function $\tau : [0, 2] \to [0, 1]$ such that

$$\tau(x) = \begin{cases} 0 & x \in [0, \bar\lambda_2] \cup [2 - \bar\lambda_2, 2], \\ 1 & x \in [\tilde\lambda_2, 2 - \tilde\lambda_2]. \end{cases} \tag{0.11}$$

Defining $z := \tau v$, then z satisfies

$$\begin{cases} z_t + (az_x)_x = h, & (t, x) \in (0, T) \times (\lambda_2, 1), \\ z(t, \lambda_2) = z(t, 1) = 0, & t \in (0, T), \end{cases}$$

with $h := (a\tau_x v)_x + a\tau_x v_x$. By applying [**2**, Theorem 3.1] with $A = \lambda_2$, $B = 1$ and related weight Φ, we have

$$\int_0^T\int_{\lambda_2}^1 \left[s\Theta(z_x)^2 e^{2s\Phi} + s^3\Theta^3 z^2 e^{2s\Phi}\right] dxdt \le c\int_0^T\int_{\lambda_2}^1 e^{2s\Phi} h^2 dxdt, \tag{0.12}$$

for $s \ge s_0$ and $c > 0$. Notice that the boundary term is nonpositive and thus is neglected. Now, proceeding as in [**2**], there exists a constant $k > 0$ such that, by [**2**, Proposition 5.2], as above, we get

$$\begin{aligned} &\int_0^T\int_{\lambda_2}^1 \Big(s\Theta a(z_x)^2 + s^3\Theta^3 \frac{(x - x_0)^2}{a} z^2\Big) e^{2s\varphi} dxdt \\ &\le k\int_0^T\int_{\lambda_2}^1 \left[s\Theta(z_x)^2 + s^3\Theta^3 z^2\right] e^{2s\Phi} dxdt \le C\int_0^T\int_\omega v^2 dxdt, \end{aligned}$$

for a positive constant C and s large enough. Hence, we get

$$\begin{aligned}(0.13)\quad &\int_0^T\int_{\tilde\lambda_2}^1 \Big(s\Theta a(v_x)^2 + s^3\Theta^3\frac{(x-x_0)^2}{a}v^2\Big)e^{2s\varphi}dxdt\\ &\le \int_0^T\int_{\lambda_2}^1 \Big(s\Theta a(z_x)^2 + s^3\Theta^3\frac{(x-x_0)^2}{a}z^2\Big)e^{2s\varphi}dxdt \le C\int_0^T\int_\omega v^2dxdt.\end{aligned}$$

Finally, consider a smooth function $\rho:[-1,1]\to[0,1]$ such that

$$(0.14)\qquad \rho(x)=\begin{cases}0 & x\in[-1,-\bar\beta_1],\\ 1 & x\in[-\tilde\lambda_1,\tilde\lambda_1],\\ 0 & x\in[\bar\beta_1,1],\end{cases}$$

and define $Z:=\rho W$, where W is defined in (0.5); thus Z satisfies (0.6) with $\bar h = (\tilde a\rho_x W)_x + \tilde a\rho_x W_x$. By applying [**2**, Theorem 3.1] with $A=-\beta_1$ and $B=\beta_1$ with the corresponding Φ, by [**2**, Propositions 5.2] we get

$$\begin{aligned}&\int_0^T\int_{-\beta_1}^{\beta_1}\left(s\Theta(Z_x)^2+s^3\Theta^3Z^2\right)e^{2s\Phi}dxdt \le c\int_0^T\int_{-\beta_1}^{\beta_1} e^{2s\Phi}\bar h^2dxdt\\ &\le C\int_0^T\int_{-\bar\beta_1}^{-\tilde\lambda_1}(W^2+(W_x)^2)e^{2s\Phi}dxdt + C\int_0^T\int_{\tilde\lambda_1}^{\bar\beta_1}(W^2+(W_x)^2)e^{2s\Phi}dxdt\\ &\le C\int_0^T\int_{-\beta_1}^{-\lambda_1}W^2dxdt + C\int_0^T\int_{\lambda_1}^{\beta_1}W^2dxdt \le C\int_0^T\int_\omega v^2dxdt,\end{aligned}$$

for some positive constants c, C and s large enough. Thus, as before, we get

$$\begin{aligned}(0.15)\quad &\int_0^T\int_0^{\tilde\lambda_1}\Big(s\Theta a(v_x)^2+s^3\Theta^3\frac{(x-x_0)^2}{a}v^2\Big)e^{2s\varphi}dxdt\\ &\le k\int_0^T\int_0^{\tilde\lambda_1}\left[s\Theta(Z_x)^2e^{2s\Phi}+s^3\Theta^3Z^2e^{2s\Phi}\right]dxdt\\ &\le k\int_0^T\int_{-\beta_1}^{\beta_1}\left(s\Theta(Z_x)^2+s^3\Theta^3Z^2\right)e^{2s\Phi}dxdt \le C\int_0^T\int_\omega v^2dxdt.\end{aligned}$$

The conclusion follows by (0.10), (0.13) and (0.15). □

The rest of the proof of Proposition 5.1 is as in [**2**].

The non divergence case

Proposition 5.3 (and consequent Theorem 5.3) holds with assumption (0.2) in place of (5.2). In particular, Lemma 5.4 holds with (0.2) in place of (5.2).

PROOF OF LEMMA 5.4. Let $\tilde\lambda_i,\bar\lambda_i,\bar\beta_i\in(\lambda_i,\beta_i)$, $i=1,2$, be as in the proof of Lemma 5.2; consider ξ, τ, ρ, $\hat\omega$ defined therein and take $w=\xi v$, $z=\tau v$ and $Z=\rho W$, where v is any fixed solution of (5.38) and W is defined in (0.5). Hence, w, z and Z satisfy

$$(0.16)\qquad \begin{cases} w_t + aw_{xx} = a(\xi_{xx}v+2\xi_xv_x) =: f, & (t,x)\in(0,T)\times(0,1),\\ w(t,0)=w(t,1)=0, & t\in(0,T),\end{cases}$$

$$\begin{cases} z_t + a z_{xx} = a(\tau_{xx} v + 2\tau_x v_x) =: h, & (t,x) \in (0,T) \times (\lambda_2, 1), \\ z(t,\lambda_2) = z(t,1) = 0, & t \in (0,T), \end{cases} \tag{0.17}$$

and

$$\begin{cases} Z_t + \tilde{a} Z_{xx} = \tilde{a}\rho_{xx} W + 2\tilde{a}\rho_x W_x = \bar{h}, & (t,x) \in (0,T) \times (-\beta_1, \beta_1), \\ Z(t,-\beta_1) = Z(t,\beta_1) = 0, & t \in (0,T). \end{cases} \tag{0.18}$$

By applying [**2**, Theorem 4.2] to (0.16), by [**2**, Proposition 5.4], we have for all $s \geq s_0$

$$\begin{aligned} &\int_0^T \int_{\tilde{\lambda}_1}^{\tilde{\lambda}_2} \left(s\Theta (v_x)^2 + s^3 \Theta^3 \left(\frac{x-x_0}{a} \right)^2 v^2 \right) e^{2s\varphi} dxdt \\ &\leq \int_{Q_T} \left(s\Theta (w_x)^2 + s^3 \Theta^3 \left(\frac{x-x_0}{a} \right)^2 w^2 \right) e^{2s\varphi}\, dxdt \leq C \int_{Q_T} \frac{e^{2s\varphi}}{a} f^2 dxdt \\ &\leq C \int_0^T \int_{\hat{\omega}} e^{2s\varphi} (v^2 + (v_x)^2) dxdt \leq C \int_0^T \int_\omega \frac{v^2}{a} xdt. \end{aligned} \tag{0.19}$$

Moreover, applying [**2**, Theorem 3.2] to (0.17) with $A = \lambda_2$, $B = 1$ and related Φ, we obtain again (0.12) with $h = a(\tau_{xx} v + 2\tau_x v_x)$.

Now, working with z, recalling what the support of τ is, we similarly get

$$\int_0^T \int_{\lambda_2}^1 \left(s\Theta (z_x)^2 + s^3 \Theta^3 z^2 \right) e^{2s\Phi} dxdt \leq C \int_0^T \int_\omega \frac{v^2}{a} dxdt. \tag{0.20}$$

As usual, we can find $k > 0$ such that for a positive constant C and s large enough

$$\begin{aligned} &\int_0^T \int_{\tilde{\lambda}_2}^1 \left(s\Theta (v_x)^2 + s^3 \Theta^3 \left(\frac{x-x_0}{a} \right)^2 v^2 \right) e^{2s\varphi} dxdt \\ &\leq k \int_0^T \int_{\lambda_2}^1 \left(s\Theta (z_x)^2 + s^3 \Theta^3 z^2 \right) e^{2s\Phi} dxdt \leq C \int_0^T \int_\omega \frac{v^2}{a} dxdt. \end{aligned} \tag{0.21}$$

Finally, applying [**2**, Theorem 3.2] to (0.18) with $A = -\beta_1$, $B = \beta_1$ and the associated Φ, we similarly get

(0.22)

$$\begin{aligned} &\int_0^T \int_0^{\tilde{\lambda}_1} \left(s\Theta a (v_x)^2 + s^3 \Theta^3 \left(\frac{x-x_0}{a} \right)^2 v^2 \right) e^{2s\varphi} dxdt \\ &\leq k \int_0^T \int_{-\beta_1}^{\beta_1} \left(s\Theta (Z_x)^2 + s^3 \Theta^3 Z^2 \right) e^{2s\Phi} dxdt \leq c \int_0^T \int_{-\beta_1}^{\beta_1} e^{2s\Phi} \bar{h}^2 dxdt \\ &\leq C \int_0^T \int_{-\bar{\beta}_1}^{-\tilde{\lambda}_1} (W^2 + (W_x)^2) e^{2s\Phi} dxdt + C \int_0^T \int_{\tilde{\lambda}_1}^{\bar{\beta}_1} (W^2 + (W_x)^2) e^{2s\Phi} dxdt \\ &\text{(by [}\mathbf{2}\text{, Propositions 5.4])} \\ &\leq C \int_0^T \int_{-\beta_1}^{-\lambda_1} W^2 dxdt + C \int_0^T \int_{\lambda_1}^{\beta_1} W^2 dxdt \leq C \int_0^T \int_\omega v^2 dxdt, \end{aligned}$$

for a positive constant C and s large enough. By (0.19), (0.21) and (0.22), the conclusion follows. □

The rest of the proof is the same as in [**2**].

3. Correction to [**1**]

The new hypotheses for [**2**] imply changes in [**1**] for Proposition 4.1 and Theorem 4.1 only if Neumann boundary conditions - shortly (Nbc) - hold, and in Lemma 5.1, while Proposition 4.1 and Theorem 4.1 are correct for Dirichlet boundary conditions - shortly (Dbc). Hypothesis 4.1 must be changed in the (Nbc) case only in the part (a_1), but for completeness we give the entire

Hypothesis 4.1 - (Nbc). (a_1) $a \in W^{1,1}(0,1)$, there exist $\sigma \in (0,1)$, $\mathfrak{g} \in L^1(-\sigma, 1+\sigma)$, $\mathfrak{h} \in W^{1,\infty}(-\sigma, 1+\sigma)$ and two strictly positive constants $\mathfrak{g}_0$, $\mathfrak{h}_0$ such that $\mathfrak{g}(x) \geq \mathfrak{g}_0$ and

$$\frac{\tilde{a}'(x)}{2\sqrt{\tilde{a}(x)}}\left(\int_x^{1+\sigma}\mathfrak{g}(t)dt + \mathfrak{h}_0\right) + \sqrt{\tilde{a}(x)}\mathfrak{g}(x) = \mathfrak{h}(x) \quad \text{for a.e. } x \in [-\sigma, 1+\sigma], \tag{0.23}$$

where

$$\tilde{a}(x) = \begin{cases} a(-x), & x \in [-1,0], \\ a(x), & x \in [0,1], \\ a(2-x), & x \in [1,2], \end{cases} \tag{0.24}$$

or
(a_2) $a \in W^{1,\infty}(0,1)$.

Set $\tilde{\Phi}(t,x) := \Theta(t)\rho_{-\sigma,1+\sigma}(x)$ and $\Phi(t,x) := \Theta(t)\rho_{0,1}(x)$, where, for $A < B$,

$$\rho_{A,B}(x) := \begin{cases} -r\left[\displaystyle\int_A^x \frac{1}{\sqrt{\tilde{a}(t)}}\int_t^B \mathfrak{g}(s)dsdt + \int_A^x \frac{\mathfrak{h}_0}{\sqrt{\tilde{a}(t)}}dt\right] - \mathfrak{c}, & \text{in the (WD)}, \\ e^{r\zeta(x)} - \mathfrak{c}, & \text{in the (SD)}, \end{cases} \tag{0.25}$$

$$\zeta(x) = \mathfrak{d}\int_x^B \frac{1}{\tilde{a}(t)}dt,$$

with $\mathfrak{d} = \|\tilde{a}'\|_{L^\infty(A,B)}$, $r > 0$ and $\mathfrak{c} > 0$ is such that $\max_{[A,B]} \rho_{A,B} < 0$.

Remark 2. (1) $\rho_{-\sigma,1+\sigma}(x) \leq \rho_{0,1}(x)$ for all $x \in [0,1]$, thus in (0.25) it is sufficient to choose $\mathfrak{c}$ such that $\max_{[0,1]} \rho_{0,1} < 0$ in order to ensure that $\rho_{-\sigma,1+\sigma} < 0$. Hence, $\tilde{\Phi}(t,x) \leq \Phi(t,x)$ for all $x \in [0,1]$,

(2) $\rho_{-\sigma,1+\sigma}(2-x) \leq \rho_{-\sigma,1+\sigma}(x) \leq \rho_{-\sigma,1+\sigma}(-x)$ for all $x \in [0,1]$. Hence,

$$\tilde{\Phi}(t,2-x) \leq \tilde{\Phi}(t,x) \leq \tilde{\Phi}(t,-x) \quad \text{for all } x \in [0,1].$$

Similar estimates hold for $\rho_{A,B}$ for any A, B.

Proposition 4.1 (Nondegenerate Carleman estimate - (Nbc)) *Assume Hypothesis 4.1 - (Nbc) case. Let z solve (4.4). Then, for any $\delta \in (0,\sigma)$ with $\delta + \sigma < 1$, there exist three positive constants $C = C(\sigma,\delta)$, r and s_0 such that for any $s \geq s_0$*

(0.26)

$$\int_{Q_T}\left(s\Theta(z_x)^2 + s^3\Theta^3 z^2\right)e^{2s\tilde{\Phi}(t,x)}dxdt \leq C\left(\int_0^T\int_0^{\sigma+\delta} z^2 dxdt + C\int_0^T\int_{1-\sigma-\delta}^1 z^2 dxdt\right.$$
$$\left. + \int_{Q_T} h^2 e^{2s\Phi(t,x)}dxdt + \int_0^T\int_0^\sigma h^2 e^{2s\tilde{\Phi}(t,-x)}dxdt\right).$$

Proof. Take W as in (4.10), fix δ as indicated and consider a smooth function $\xi : [-1,2] \to \mathbb{R}$ such that $\xi \equiv 1$ in $[-\delta, 1+\delta]$ and $\xi \equiv 0$ in $[-1,-\sigma] \cup [1+\sigma, 2]$. Set $Z := \xi W$, so that Z solves

$$\begin{cases} Z_t + \tilde{a} Z_{xx} + \lambda \dfrac{Z}{\tilde{b}} = H, & (t,x) \in (0,T) \times (-\sigma, 1+\sigma), \\ Z(t,-\sigma) = Z(t, 1+\sigma) = 0, & t \in (0,T), \end{cases}$$

with $H := \xi \tilde{h} + \tilde{a}(\xi_{xx} W + 2\xi_x W_x)$, $\tilde{b}$ and $\tilde{h}$ defined in (4.12) and (4.13), respectively. Thus, we can apply [**1**, Proposition 4.1] in the (Dbc) case with $(-\sigma, 1+\sigma)$ in place of $(0,1)$, $\tilde{a}$ in place of a and with $Z_x(t,-\sigma) = Z_x(t, 1+\sigma) = 0$. Thus, [**1**, Hypothesis 4.1] holds in the (Dbc) case. Hence, we find two positive constants C (depending on σ) and s_0 (s_0 sufficiently large), such that Z satisfies, for all $s \geq s_0$

(0.27)

$$\int_0^T \int_{-\sigma}^{1+\sigma} \left(s\Theta (Z_x)^2 + s^3\Theta^3 Z^2\right) e^{2s\tilde{\Phi}} dxdt \leq C \int_0^T \int_{-\sigma}^{1+\sigma} H^2 e^{2s\tilde{\Phi}} dxdt$$
$$\leq C \int_0^T \int_{-\sigma}^{1+\sigma} \tilde{h}^2 e^{2s\tilde{\Phi}} dxdt + C \left(\int_0^T \int_{-\sigma}^{-\delta} (W^2 + W_x^2) e^{2s\tilde{\Phi}} dxdt + \int_0^T \int_{1+\delta}^{1+\sigma} (W^2 + W_x^2) e^{2s\tilde{\Phi}} dxdt \right)$$

(by [**1**, Remark 7])

$$\leq C \int_0^T \int_{-\sigma}^{1+\sigma} \tilde{h}^2 e^{2s\tilde{\Phi}} dxdt + C \int_0^T \int_0^{\sigma+\delta} z^2 dxdt + C \int_0^T \int_{1-\sigma-\delta}^{1} z^2 dxdt.$$

Now, observe that

(0.28)

$$\int_0^T \int_{-\sigma}^{1+\sigma} \tilde{h}^2(t,x) e^{2s\tilde{\Phi}(t,x)} dxdt \leq \int_0^T \int_0^{\sigma} h^2(t,x) e^{2s\tilde{\Phi}(t,-x)} dxdt + \int_{Q_T} h^2(t,x) e^{2s\Phi(t,x)} dxdt.$$

Indeed, using a change of variable, the definition of $\tilde{h}$ and Remark 2, we get

$$\int_{-\sigma}^{1+\sigma} \tilde{h}^2(t,x) e^{2s\tilde{\Phi}(t,x)} dx = \int_{-\sigma}^{0} \tilde{h}^2(t,x) e^{2s\tilde{\Phi}(t,x)} dx + \int_0^1 \tilde{h}^2(t,x) e^{2s\tilde{\Phi}(t,x)} dx + \int_1^{1+\sigma} \tilde{h}^2(t,x) e^{2s\tilde{\Phi}(t,x)} dx$$
$$= \int_0^{\sigma} \tilde{h}^2(t,-y) e^{2s\tilde{\Phi}(t,-y)} dy + \int_0^1 h^2(t,x) e^{2s\tilde{\Phi}(t,x)} dx + \int_{1-\sigma}^{1} \tilde{h}^2(t, 2-y) e^{2s\tilde{\Phi}(t,2-y)} dy$$
$$\leq \int_0^{\sigma} h^2(t,y) e^{2s\tilde{\Phi}(t,-y)} dy + \int_0^1 h^2(t,x) e^{2s\Phi(t,x)} dx + \int_{1-\sigma}^{1} h^2(t,y) e^{2s\Phi(t,y)} dy.$$

Thus, by (0.27) and (0.28), we obtain

(0.29)

$$\int_0^T \int_{-\sigma}^{1+\sigma} \left(s\Theta (Z_x)^2 + s^3\Theta^3 Z^2\right) e^{2s\tilde{\Phi}} dxdt \leq C \left(\int_0^T \int_0^{\sigma+\delta} z^2 dxdt + \int_0^T \int_{1-\sigma-\delta}^{1} z^2 dxdt \right)$$
$$+ C \int_0^T \int_0^{\sigma} h^2(t,x) e^{2s\tilde{\Phi}(t,-x)} dxdt + C \int_{Q_T} h^2(t,x) e^{2s\Phi(t,x)} dxdt.$$

As a consequence,

$$\int_{Q_T}\left(s\Theta(z_x)^2+s^3\Theta^3z^2\right)e^{2s\tilde\Phi}dxdt\le\int_0^T\int_{-\sigma}^{1+\sigma}\left(s\Theta(Z_x)^2+s^3\Theta^3Z^2\right)e^{2s\tilde\Phi}dxdt$$
$$\le C\left(\int_0^T\int_0^{\sigma+\delta}z^2dxdt+\int_0^T\int_{1-\sigma-\delta}^1 z^2dxdt\right.$$
$$\left.+\int_0^T\int_0^{\sigma}h^2(t,x)e^{2s\tilde\Phi(t,-x)}dxdt+\int_{Q_T}h^2(t,x)e^{2s\Phi(t,x)}dxdt\right),$$

for a positive constant C depending on σ and δ. The claim follows. □

REMARK 3. Proposition 4.1 still holds if we replace $[0,1]$ with a general interval $[A,B]$ where a and b satisfy Hypothesis 4.1 in $[A-\sigma,B+\sigma]$. In this case $\int_0^T\int_0^\sigma h^2(t,x)e^{2s\tilde\Phi(t,-x)}dxdt$ is replaced by $\int_0^T\int_A^{A+\sigma}h^2(t,x)e^{2s\tilde\Phi(t,2A-x)}dxdt$ with $\tilde\Phi:=\Theta(t)\rho_{A-\sigma,B+\sigma}$.

Again, in the weak degenerate case, there is a change only in the (Nbc) case:

Hypothesis 4.2 - (Nbc). The assumption is as in [**1**] and in addition: if Hypothesis 1.1 or Hypothesis 1.3 of [**1**] holds, there exist $B_1\in(0,x_0)$, $B_2\in(1,2-x_0)$, two functions $\mathfrak{g}\in L^\infty_{\text{loc}}((-x_0,2-x_0)\setminus\{x_0\})$, $\mathfrak{h}(\cdot,B_i)\in W^{1,\infty}_{\text{loc}}((-x_0,2-x_0)\setminus\{x_0\})$ and two strictly positive constants $\mathfrak{g}_0$, $\mathfrak{h}_0$ such that $\mathfrak{g}(x)\ge\mathfrak{g}_0$ and

$$\frac{\tilde a'(x)}{2\sqrt{\tilde a(x)}}\left(\int_x^{B_i}\mathfrak{g}(t)dt+\mathfrak{h}_0\right)+\sqrt{\tilde a(x)}\mathfrak{g}(x)=\mathfrak{h}(x,B_i)\tag{0.30}$$

with $i=1,2$, $-x_0<x<B_1$ or $x_0<x<B_2$, and $\tilde a$ defined as in (0.24).

Using $\varphi:=\Theta\psi$ and the space $\mathcal{V}$ introduced in [**1**], Theorem 4.1 in the (Nbc) is improved in this way:

Theorem 4.1 - (Nbc). *Assume Hypothesis* 4.2 - *(Nbc) . Let $\omega\subset\subset(0,1)$ be an open interval containing x_0, B_1 and $2-B_2$, or let $\omega=\omega_1\cup\omega_2$, where $\omega_i=(\lambda_i,\beta_i)\subset(0,1)$, $i=1,2$, $\beta_1\le B_1$ and $2-B_2\le\lambda_2$. Then, there exist two positive constants C and s_0 (depending on λ) such that every solution $v\in\mathcal{V}$ of* (4.26) *satisfies, for all $s\ge s_0$,*

(0.31)
$$\int_{Q_T}\left(s\Theta(v_x)^2+s^3\Theta^3\left(\frac{x-x_0}{a}\right)^2v^2\right)e^{2s\varphi}dxdt\le C\left(\int_0^T\int_\omega v^2dxdt+\int_{Q_T}\frac{h^2(t,x)}{a}e^{2s\varphi}dxdt\right)$$
$$+C\left(\int_0^T\int_0^{B_1}\frac{h^2(t,x)}{a}e^{2s\Phi_1(t,-x)}dxdt+\int_0^T\int_{2-B_2}^1\frac{h^2(t,x)}{a}e^{2s\Phi_2(t,x)}dxdt\right),$$

where $\Phi_1(t,x):=\Theta(t)\rho_{-B_1,B_1}(x)$ and $\Phi_2(t,x):=\Theta(t)\rho_{2-B_2,B_2}(x)$.

PROOF. First, assume that $x_0,B_1,2-B_2\in\omega$. Then, we can fix two subintervals $\omega_1=(\lambda_1,\beta_1)\subset\subset(0,x_0),\omega_2=(\lambda_2,\beta_2)\subset\subset(x_0,1)$, with $\beta_1=B_1$ and $\lambda_2=2-B_2$. Let v solve (4.26), take ξ as in (0.8) and define $w:=\xi v$, so that w satisfies

$$\begin{cases}w_t+a(x)w_{xx}+\dfrac{\lambda}{b(x)}w=\bar h, & (t,x)\in Q_T,\\ w(t,0)=w(t,1)=0, & t\in(0,T),\end{cases}\tag{0.32}$$

where $\bar h := \xi h + a(\xi_{xx} v + 2\xi_x v_x)$. Using [**1**, Theorem 4.1] in the (Dbc) case, one has

$$\int_0^T\int_0^1 \left(s\Theta a (w_x)^2 + s^3\Theta^3\left(\frac{x-x_0}{a}\right)^2 w^2\right)e^{2s\varphi}\,dxdt \le C\int_0^T\int_0^1 \frac{\bar h^2}{a}e^{2s\varphi}dxdt, \tag{0.33}$$

for all $s \ge s_0$. Hence, by the Caccioppoli inequality [**1**, Proposition 5.2], proceeding as in (0.10), we find

$$\begin{aligned}&\int_0^T\int_{\tilde\lambda_1}^{\tilde\lambda_2}\left(s\Theta a(v_x)^2 + s^3\Theta^3\left(\frac{x-x_0}{a}\right)^2 v^2\right)e^{2s\varphi}dxdt\\ &\le C\int_0^T\int_0^1 \frac{h^2}{a}e^{2s\varphi}dxdt + C\int_0^T\int_\omega v^2 dxdt.\end{aligned} \tag{0.34}$$

Defining $z := \tau W$, W being defined in (4.25) and τ in (0.11), then z satisfies

$$\begin{cases} z_t + \tilde a(x) z_{xx} + \dfrac{\lambda}{\tilde b(x)} z = f, & (t,x)\in(0,T)\times(\lambda_2, 2-\lambda_2)\\ z(t,\lambda_2) = z(t, 2-\lambda_2) = 0, & t\in(0,T),\end{cases} \tag{0.35}$$

with $f := \tau\tilde h + \tilde a(\tau_{xx}W + 2\tau_x W_x)$. Then, [**1**, Proposition 4.1] in the (Dbc) case can be applied; by Remark 2 and the Caccioppoli inequality in the non degenerate case [**1**, Remark 7], we get

$$\begin{aligned}&\int_0^T\int_{\lambda_2}^{2-\lambda_2}\left[s\Theta(z_x)^2e^{2s\Phi_2} + s^3\Theta^3 z^2 e^{2s\Phi_2}\right]dxdt \le C\int_0^T\int_{\lambda_2}^{2-\lambda_2} f^2(t,x)e^{2s\Phi_2(t,x)}dxdt\\ &\le C\int_0^T\int_{\lambda_2}^1 \frac{h^2(t,x)}{a}e^{2s\Phi_2(t,x)}dxdt + C\int_0^T\int_{\tilde\omega}(W^2+W_x^2)e^{2s\Phi_2(t,x)}dxdt\\ &\le C\int_0^T\int_{\lambda_2}^1 \frac{h^2(t,x)}{a}e^{2s\Phi_2(t,x)}dxdt + C\int_0^T\int_\omega v^2dxdt,\end{aligned} \tag{0.36}$$

where $\tilde\omega = (\bar\lambda_2, \tilde\lambda_2)\cup(2-\tilde\lambda_2, 2-\bar\lambda_2)$. Proceeding as before, there exists $k>0$ such that, for a positive constant C and s large enough,

$$\begin{aligned}&\int_0^T\int_{\lambda_2}^1\left(s\Theta(z_x)^2 + s^3\Theta^3\left(\frac{x-x_0}{a}\right)^2 z^2\right)e^{2s\varphi}dxdt\\ &\le k\int_0^T\int_{\lambda_2}^{2-\lambda_2}\left[s\Theta(z_x)^2e^{2s\Phi_2} + s^3\Theta^3z^2e^{2s\Phi_2}\right]dxdt\\ &\le C\int_0^T\int_{\lambda_2}^1\frac{h^2(t,x)}{a}e^{2s\Phi_2(t,x)}dxdt + C\int_0^T\int_\omega v^2dxdt.\end{aligned}$$

Hence, by definition of z and by the inequality above, we get

$$\begin{aligned}&\int_0^T\int_{\tilde\lambda_2}^1\left(s\Theta(v_x)^2 + s^3\Theta^3\left(\frac{x-x_0}{a}\right)^2 v^2\right)e^{2s\varphi}dxdt\\ &= \int_0^T\int_{\tilde\lambda_2}^1\left(s\Theta(z_x)^2 + s^3\Theta^3\left(\frac{x-x_0}{a}\right)^2 z^2\right)e^{2s\varphi}dxdt\\ &\le C\int_0^T\int_{\lambda_2}^1\frac{h^2(t,x)}{a}e^{2s\Phi_2(t,x)}dxdt + C\int_0^T\int_\omega v^2dxdt.\end{aligned} \tag{0.37}$$

Finally, defining $Z := \rho W$, where ρ is given in (0.14), then Z satisfies

$$\begin{cases} Z_t + \tilde{a} Z_{xx} + \lambda \dfrac{Z}{\tilde{b}} = \bar{h}, & (t,x) \in (0,T) \times (-\beta_1, \beta_1), \\ Z(t,-\beta_1) = Z(t,\beta_1) = 0, & t \in (0,T), \end{cases} \tag{0.38}$$

with $\bar{h} := \rho \tilde{h} + \tilde{a}(\rho_{xx} W + 2\rho_x W_x)$. Proceeding as before, by [**1**, Proposition 4.1] in the (Dbc) case, Remark 2 and [**1**, Remark 7], one has

$$\begin{aligned} &\int_0^T \int_0^{\tilde{\lambda}_1} \left(s\Theta (v_x)^2 + s^3 \Theta^3 \left(\frac{x - x_0}{a} \right)^2 v^2 \right) e^{2s\varphi} dxdt \\ &\le C \int_0^T \int_{-\beta_1}^{\beta_1} \left[s\Theta (Z_x)^2 e^{2s\Phi_1} + s^3 \Theta^3 Z^2 e^{2s\Phi_1} \right] dxdt \\ &\le C \int_0^T \int_0^{\beta_1} \frac{h^2(t,x)}{a} e^{2s\Phi_1(t,-x)} dxdt + C \int_0^T \int_\omega v^2 dxdt. \end{aligned} \tag{0.39}$$

The conclusion follows by (0.10) and (0.37).

Nothing changes in the proof if $\omega = \omega_1 \cup \omega_2$ and each of these intervals lye on different sides of x_0, as the assumption implies. □

3.1. Applications to observability inequality. In this subsection we shall make the necessary changes according to the new hypotheses. For this, we assume that the control set ω satisfies the following assumption:

Hypothesis 5.1. (0.1) or (0.2) hold.

Then, [**1**, Hypothesis 5.2] reads:

Hypothesis 5.2 Hypothesis 4.2 of [**1**] is satisfied. Moreover, under Hypothesis 1.1 or Hypothesis 1.3, condition (0.30) holds with $B_1 = \beta_1$ and $B_2 = 2 - \lambda_2$.

With this changes, Proposition 5.1 still holds true and the only changes are in the proof of Lemma 5.1.

Proof of Lemma 5.1 if (0.2) holds. As in the proof of Lemma 5.2 in Section 2 above, fix $\tilde{\lambda}_i, \bar{\lambda}_i, \bar{\beta}_i \in (\lambda_i, \beta_i)$, $i = 1, 2$, and smooth functions ξ, ρ, τ as given in (0.8), (0.14), (0.11). Set $w := \xi v$, so that w satisfies

$$\begin{cases} w_t + a w_{xx} + \lambda \dfrac{w}{b} = a(\xi_{xx} v + 2\xi_x v_x) =: f, & (t,x) \in Q_T, \\ w(t,0) = w(t,1) = 0, & t \in (0,T). \end{cases}$$

Applying [**1**, Theorem 4.1] with (Dbc), there exists two positive constants C and s_0 such that, for all $s \ge s_0$,

$$\int_{Q_T} \left(s\Theta (w_x)^2 + s^3 \Theta^3 \left(\frac{x - x_0}{a} \right)^2 w^2 \right) e^{2s\varphi} \, dxdt \le C \int_{Q_T} \frac{e^{2s\varphi}}{a} f^2 dxdt. \tag{0.40}$$

Then, (0.19) holds.

Now, set $z := \tau W$, where W is defined in (0.5) or in (4.25) if (Dbc) or (Nbc) hold, respectively, so that z satisfies (0.35) with $\tilde{h} = 0$. Then (0.37) holds with $h = 0$.

Finally, set $Z := \rho W$, so that Z satisfies (0.38) with $\bar{h} = \tilde{a}(\rho_{xx} W + 2\rho_x W_x)$. By applying [**1**, Proposition 4.1] in $(-\beta_1, \beta_1)$ in the (Dbc) case, together with [**1**, Remark 7], we get

$$\int_0^T \int_{-\beta_1}^{\beta_1} \left(s\Theta (Z_x)^2 + s^3 \Theta^3 Z^2 \right) e^{2s\Phi_1} dxdt$$
$$\leq C \int_0^T \int_{-\bar{\beta}_1}^{-\tilde{\lambda}_1} (W^2 + (W_x)^2) e^{2s\Phi_1} dxdt + C \int_0^T \int_{\tilde{\lambda}_1}^{\bar{\beta}_1} (W^2 + (W_x)^2) e^{2s\Phi_1} dxdt$$
$$\leq C \int_0^T \int_{-\beta_1}^{-\lambda_1} W^2 dxdt + C \int_0^T \int_{\lambda_1}^{\beta_1} W^2 dxdt \leq C \int_0^T \int_\omega v^2 dxdt \leq C \int_0^T \int_\omega \frac{v^2}{a} dxdt,$$

for some positive constants c, C and s large enough. Hence, (0.39) holds with $h = 0$. Therefore, by (0.19) and (0.37), (0.39) with $h = 0$, the conclusion follows. □

Proof of Lemma 5.1 if (0.1) holds. Let ω_1, ω_2 be as in Remark 1. Then (5.27) holds. Now, take $z = \tau W$, with τ as in (0.11) and W as in the previous case; hence, z satisfies (0.35) with $f = \tilde{a}(\tau_{xx} W + 2\tau_x W_x)$, and so (0.37) holds with $h = 0$.

Finally, we consider $Z := \rho W$ with ρ as in (0.14). Then, Z satisfies (0.38) with $\bar{h} = \tilde{a}(\rho_{xx} W + 2\rho_x W_x)$. Thus, as above, (0.39) holds with $h = 0$. Hence, by (5.27) and (0.37), (0.39) with $h = 0$, the conclusion follows. □

4. Correction to [3]

Here Hypothesis 4.1(ii) is replaced by (0.2), and so Lemma 4.7 and Proposition 4.4 still hold true.

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